Exceptional Operators in N=4 SYM

Gleb Arutyunov

Institute for Theoretical Physics
Utrecht University

based on hep-th/1205.6660 with S. Frolov and A Sfondrini

$$\mathcal{O}_L = \sum_{i=1}^{L-4} (-1)^i \,\mathrm{tr}\Big(X X\, Z^i X\, Z^{L-i-3} \Big)$$

$$1 = e^{ip_k L} \prod_{j \neq k}^{M} S_{\mathrm{xxx}}(u_k, u_j) \quad \Rightarrow \quad 1 = \left(\frac{u_k + i}{u_k - i} \right)^L \prod_{j \neq k}^{M} \frac{u_k - u_j - 2i}{u_k - u_j + 2i}\,, \quad k = 1, \ldots, M\,.$$

$$e^{iP} = 1 \quad \Leftrightarrow \quad \prod_{k=1}^{M} \frac{i + u_k}{i - u_k} = 1\,, \qquad P = \sum_{k=1}^{M} p(u_k),$$

$$E = L + g^2 \sum_{k=1}^{M} \frac{2}{1 + u_k^2}$$

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$$u_1 = 0\,, \qquad u_2 = -i\,, \qquad u_3 = i\,,$$

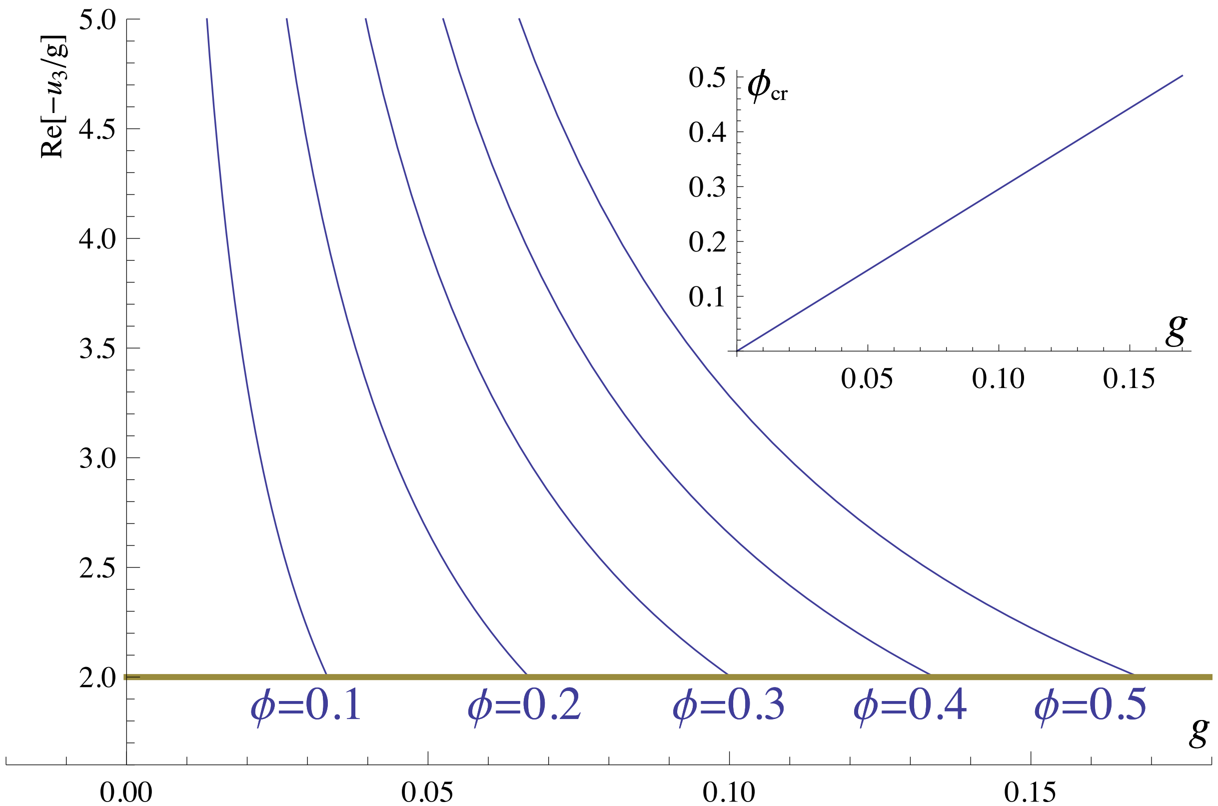
$$p_1 = \pi, \qquad p_2 = -\frac{\pi}{2} + i\,\infty, \qquad p_3 = -\frac{\pi}{2} - i\,\infty$$

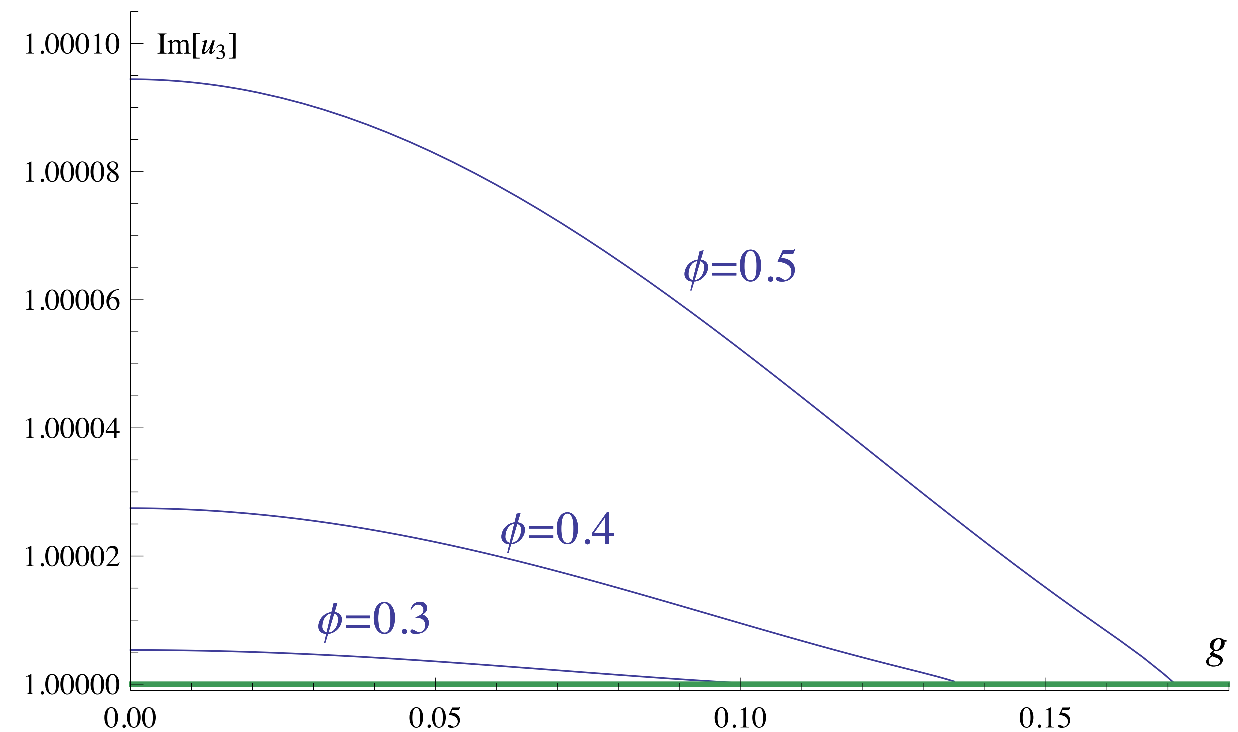
$$1 = e^{-i\phi} \left(\frac{u_k + i}{u_k - i}\right)^L \prod_{j \neq k}^{M} \frac{u_k - u_j - 2i}{u_k - u_j + 2i}$$

$$u_1 \sim \phi\,, \qquad u_2 \sim -i - \phi - i\,\phi^L\,, \qquad u_3 \sim +i - \phi + i\,\phi^L$$

$$\lim_{\phi \to 0} E(\phi) = L + 3g^2$$

$$1 = e^{-i\phi} e^{ip_k L} \prod_{j \neq k}^{M} \frac{u_k - u_j - 2i}{u_k - u_j + 2i} \sigma^{-2}(u_k, u_j)$$





$$u_i = \sum_{n=0}^{L-1} f_{i,n}(\phi, L)\, g^{2n} + \mathcal{O}(g^{2L})$$

$$E^{\rm asym} = J + \sum_{k=1}^{M} \sqrt{1 + 4g^2\ \sin^2(p_k/2)}$$

$$E^{\rm asym} = 6 + 3g^2 - \frac{9}{4}g^4 + \frac{63}{16}g^6 - \frac{621}{64}g^8 - \frac{9}{256}(8\zeta(3) - 783)g^{10} + \\ +\Big(- \frac{2187}{1024\phi^6} - \frac{3645}{8192\phi^4} + \frac{189783}{1310720\phi^2} + \frac{81}{128}\zeta(5) + \frac{27}{32}\zeta(3) - \frac{1223982387}{14680064}\Big)g^{12}$$

$$g \lesssim \phi \ll 1$$

$$E = J + \sum_{i=1}^{3} \mathcal{E}(u_i^{(1)}) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}_Q}{du} \log(1 + Y_Q)$$
$$-i\tilde{p}_2(u_2^{(1)+}) + i\tilde{p}_2(u_2^{(2)+}) - i\tilde{p}_2(u_3^{(2)-}) + i\tilde{p}_2(u_3^{(1)-})$$

$$1 + Y_2(u_2^{(2)+}) = 0 \qquad 1 + Y_2(u_3^{(2)-}) = 0 \qquad Y_2(u_2^{(1)+}) = \infty\,, \qquad Y_2(u_3^{(1)-}) = \infty$$

$$\Delta E^{(\mathrm{wrap})} = -\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du\, \frac{d\tilde{p}_Q}{du}\, Y_Q^{\circ}$$
$$-i \frac{\partial \tilde{p}_2}{\partial u}(u_2^{(1)+})\, \mathrm{Res} Y_2^{\circ}\big(u_2^{(1)+}\big) + i \frac{\partial \tilde{p}_2}{\partial u}(u_3^{(1)-})\, \mathrm{Res} Y_2^{\circ}\big(u_3^{(1)-}\big)$$

$$E = 6 + 3g^2 - \frac{9}{4}g^4 + \frac{63}{16}g^6 - \frac{621}{64}g^8 - \frac{9}{256}(8\zeta(3) - 783)g^{10} + \left(-\frac{567}{128}\zeta(9) + \frac{189}{64}\zeta(5) + \frac{243}{128}\zeta(3) - \frac{84753}{1024}\right)g^{12} + \mathcal{O}(g^{14}, \phi)$$

TBA with exceptional rapidities

$$u_1 = +\frac{i0}{2}\,, \quad u_2 = -\frac{i}{g} - i0\,, \quad u_3 = \frac{i}{g} - i0$$

Y^o-function	Zeroes	Poles
$Y_{M\vert w}$	0^2	
$1+Y_{M\vert w}$	$-i/g\,,\ +i/g$	$-(M+2)i/g\,,\ (M+2)i/g$
$Y_{1\vert vw}$	0^2	
$1+Y_{M\vert vw}$		$Mi/g\,,\ -Mi/g$
Y_-	$-2i/g\,,\ 2i/g$	0^2
Y_+		$0^2\,,\ -i/g$
$1-Y_-$	$-i/g\,,\ i/g$	
$1-Y_+$		
Y_1	0^2	$-i/g\,,\ +i/g$
Y_2		0^2
$Y_Q\,, Q\geq 3$		$i(Q-2)/g\,,\ -i(Q-2)/g$

Simplified equations for $Y_{M|w}$

$$\log Y_{M|w} = 2\log S(\frac{i}{g}+v)+\log(1+Y_{M-1|w})(1+Y_{M+1|w})\star s+\delta_{M1}\ \log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}}\,\hat{\star}\, s\,.$$

Simplified equations for $Y_{M|vw}$

$$\begin{aligned}\log Y_{M|vw} =& 2\delta_{M1}\log S(\frac{i}{g}+v)+\log(1+Y_{M-1|vw})(1+Y_{M+1|vw})\star s\\ &+\delta_{M1}\log\frac{1-Y_-}{1-Y_+}\,\hat{\star}\, s-\log(1+Y_{M+1})\star s\,.\end{aligned}$$

Simplified equations for $Y_\pm$

$$\begin{aligned}\log\frac{Y_+}{Y_-} =&\ \log(1+Y_Q)\star K_{Qy}-\sum_i\log S_{1_*y}(u_i,v)\,,\\ \log Y_+Y_- =&\ 2\log\frac{1+Y_{1|vw}}{1+Y_{1|w}}\star s-\log\left(1+Y_Q\right)\star K_Q+2\log(1+Y_Q)\star K^{Q1}_{xv}\star s\\ &-4\log S(\frac{i}{g}+v)-\sum_i\log\frac{S^{1_*1}_{xv}(u_i,v)^2}{S_2(u_i-v)}\star s\,.\end{aligned}$$

$$
\begin{aligned}
G_Q(v) \;=&\; -L_{\rm TBA}\,\widetilde{\mathcal{E}}_Q + \log\big(1+Y_{Q'}\big)\star(K_{\mathfrak{sl}(2)}^{Q'Q} + 2s\star K_{vwx}^{Q'-1,Q})\\
&+\; 2\log\big(1+Y_{1|vw}\big)\star s\,\hat{\star}\,K_{yQ} + 2\log(1+Y_{Q-1|vw})\star s\\
&-\; 2\log\frac{1-Y_-}{1-Y_+}\,\hat{\star}\,s\star K_{vwx}^{1Q} + \log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}}\,\hat{\star}\,K_Q + \log\big(1-\frac{1}{Y_-}\big)\big(1-\frac{1}{Y_+}\big)\,\hat{\star}\,K_{yQ}
\end{aligned}
$$

$$
\begin{aligned}
\log Y_Q(v) = G_Q(v) - \sum_i \log S_{\mathfrak{sl}(2)}^{1_*Q}(u_i,v) + 4\log S\star_{p.v.} K_{vwx}^{1Q}(-\frac{i}{g},v)\\
-\log S_Q(-\frac{i}{g}-v)S_{yQ}(-\frac{i}{g},v)S_Q(-v)S_{yQ}(0,v)S_Q(\frac{2i}{g}-v)S_{yQ}(\frac{2i}{g},v)
\end{aligned}
$$

Exact Bethe equations

$$
Y_{1_*}(0) = -1\,,\quad Y_{1_*}(-\frac{i}{g}) = -1\,,\quad Y_{1_*}(\frac{i}{g}) = -1
$$

Scaling dimensions of exceptional operators

$$\Delta - J = E - J = \sum_i \mathcal{E}(u_i) - \frac{1}{2\pi}\int_{-\infty}^{\infty} du \frac{d\tilde{p}_Q}{du}\log(1+Y_Q)$$
$$= \sqrt{1+4g^2} + \sqrt{4+4g^2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} du \frac{d\tilde{p}_Q}{du}\log(1+Y_Q)$$

$$E_{\phi=0} = -\frac{9g^{10}\zeta(3)}{32} + \frac{7047g^{10}}{256} - \frac{621g^8}{64} + \frac{63g^6}{16} - \frac{9g^4}{4} + 3g^2 + 6$$

$$E_{\mathrm{asym}} = \sqrt{1+4g^2} + \sqrt{4+4g^2} \approx \frac{3591g^{10}}{128} - \frac{645g^8}{64} + \frac{33g^6}{8} - \frac{9g^4}{4} + 3g^2 + 6\,.$$

$$E = E_{\mathrm{asym}} - \frac{1}{2\pi}\int du \frac{d\tilde{p}_2}{du}\log(1+Y_2)$$

$$Y_2(u) = \frac{9g^{12}\left(3g^4(8\zeta(3)+15)-24g^2+8\right)}{2048u^2} + const + \mathcal{O}(u^2)$$

$$E^{pole} = -\frac{1}{2\pi}\int dv \frac{d\tilde{p}_2}{dv}\log(1+Y_2) = -\frac{9g^{10}\zeta(3)}{32} - \frac{135g^{10}}{256} + \frac{3g^8}{8} - \frac{3g^6}{16}$$

$$Y_2(u) = \frac{9g^{12}\left(3g^4(8\zeta(3)+15) - 24g^2 + 8\right)}{2048u^2} + const + \mathcal{O}(u^2)$$

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Next-to-leading TBA correction at g^{12}

$$Y(u) = Y^{\circ}(u)\left(1 + \mathscr{Y}(u)\right)$$

at the g^6 order

$$\begin{aligned}\mathscr{Y}_2 &= \log(1+Y_2^{\circ}) \star (K^{22}_{\mathfrak{sl}(2)} + 2s \star K^{12}_{vwx}) + 4\left(A_{1|vw}\,\mathscr{Y}_{1|vw}\right) \star s\,,\\ \mathscr{Y}_{M|vw} &= A_{M-1|vw}\mathscr{Y}_{M-1|vw} \star s + A_{M+1|vw}\mathscr{Y}_{M+1|vw} \star s - \delta_{M1}\log(1+Y_2^{\circ}) \star s\end{aligned}$$

$$A_{M|vw} = \frac{Y^{\circ}_{M|vw}}{1+Y^{\circ}_{M|vw}}$$

$$E^{(12)}_{\text{asym}} = -\frac{43029 g^{12}}{512}$$

$$E_Y^{(Q\neq 2)} = -\frac{1}{2\pi}\sum_{Q\neq 2}\int du\, Y_Q^{\circ}$$

$$E_Y^{(Q\neq 2)} = g^{12}\left(\frac{135\zeta(3)}{128} + \frac{297\zeta(5)}{128} - \frac{567\zeta(9)}{128} + \frac{358424597369}{580608000000}\right)$$

$$E^{(12)} - E^{(12)}_{\phi=0} = \frac{3}{512} g^{12} (12\mathscr{X}_1(0) + 7 - 12\log 2)$$

$$\frac{\mathscr{X}_M(u+i) + \mathscr{X}_M(u-i)}{A_{M|vw}(u)} = \mathscr{X}_{M-1} + \mathscr{X}_{M+1} + \delta_{M1}\, 2\pi s(u)$$

$$\mathscr{X}_1(0) = \log 2 - \frac{7}{12} \approx 0.109814$$

Relativistic dispersion relation: $\mathrm{H}^2 - p^2 = 1$
Rapidity parametrization: $\mathrm{H} = \cosh\theta, p = \sinh\theta$

Dispersion:

$$\mathrm{H}^2 - 4g^2 \sin^2 \frac{p}{2} = 1$$

uniformizes on the elliptic curve

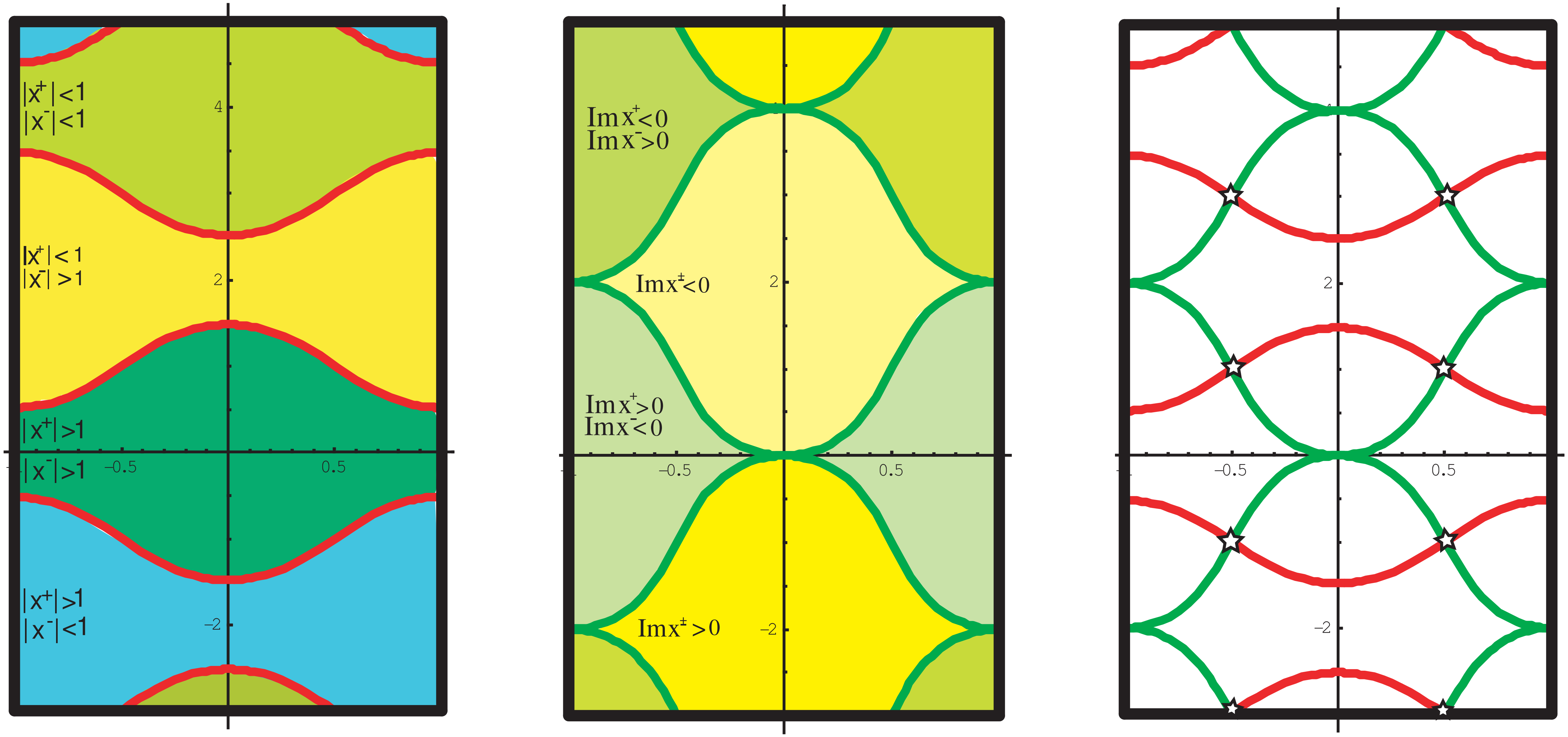
$$p = 2\,\mathrm{am}\, z\,, \quad sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \quad H = \mathrm{dn}(z,k)\,, \quad k = -4g^2$$

Useful parametrizations

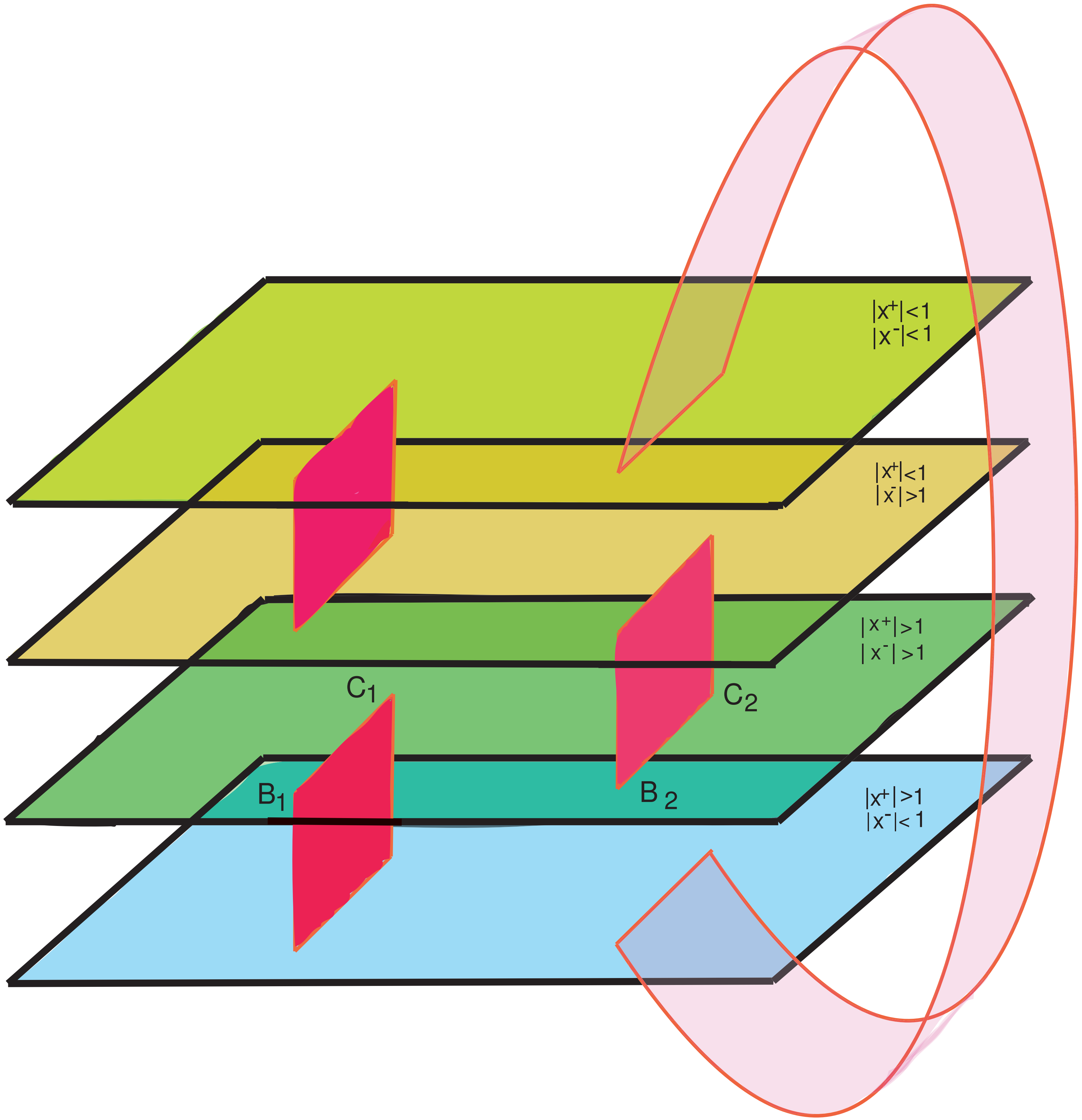
$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}$$

$$u = x^+ + \frac{1}{x^+} - \frac{i}{g} = x^- + \frac{1}{x^-} + \frac{i}{g}$$

Rapidity Torus



$$x = \mathrm{Re}(\frac{2}{\omega_1} z)\,, \quad y = \mathrm{Re}(\frac{4}{\omega_2} z)$$



Gluing torus out of four planes

$$\phi\, e^{-\lambda/(\phi-\phi_{\mathrm{cr}}(\lambda))^2}$$