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# Extremal Black Holes, Nilpotent Orbits and Tits Satake Universality classes

Based on work in collaboration with A. Sorin

# D=4 Supergravity

$$\mathcal{L}^{(4)} = \sqrt{|\det g|}\left[\frac{R[g]}{2} - \frac{1}{4}\partial_\mu\phi^a\partial^\mu\phi^b h_{ab}(\phi) + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma|\mu\nu}\right]$$
$$+\frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma},$$

$$\mathcal{M}^{D=4}_{scalar} = \frac{\mathrm{U}_{\mathrm{D=4}}}{\mathrm{H}_c}$$

Under rotating extremal black holes

$$ds^2 = -\exp[U]\left(dt + \mathbf{A}^{[KK]}\right)^2 + \exp[-U]\,dx^i \otimes dx^j\,\delta_{ij}$$

**Generic Static Solutions**

$$ds^2 = -\exp[U]\left(dt + \mathbf{A}^{[KK]}\right)^2 + \exp[-U]\,dx^i \otimes dx^j\,\gamma_{ij}(x)$$

# Sigma model representation of static solutions

$$
\begin{aligned}
\mathcal{A}^{[3]} &= \int \sqrt{\det\gamma}\,\mathfrak{R}[\gamma]\, d^3x + \int \sqrt{\det\gamma}\,\mathcal{L}^{(3)}\, d^3x \\
\mathcal{L}^{(3)} &= \Big(\partial_i U\,\partial_j U + h_{rs}\,\partial_i\phi^r\,\partial_j\phi^s \\
&\quad + e^{-2U}\left(\partial_i a + \mathbf{Z}^T\mathbb{C}\partial_i\mathbf{Z}\right)\left(\partial_j a + \mathbf{Z}^T\mathbb{C}\partial_j\mathbf{Z}\right) + 2\,e^{-U}\,\partial_i\mathbf{Z}^T\,\mathcal{M}_4\,\partial_j\mathbf{Z}\Big)\,\gamma^{ij}
\end{aligned}
$$

D=3 scalars

| | Generic | | $\mathcal{N}=2$ |
|---|---|---|---|
| warp factor | $U(x)$ | 1 | 1 |
| Taub Nut field | $a(x)$ | 1 | 1 |
| D=4 scalars | $\phi^a(x)$ | $n_s$ | $2n$ |
| Scalars from vectors | $Z^M(x) = \left(Z^\Lambda(x)\,,\,Z_\Sigma(x)\right)$ | $2n_V$ | $2n+2$ |
| **Total** | | $2+n_s+2n_V$ | $4n+4$ |

$$
\mathcal{M}_4 = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \\ \hline \mathrm{Re}\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \end{array}\right)
$$

New coset manifold with Lorentzian signature

$$
\mathcal{Q} = \frac{\mathrm{U}_{D=3}}{\mathrm{H}^\star}
$$

# Oxidation

$$
\begin{aligned}
\mathbf{F}^{[KK]} &= d\mathbf{A}^{[KK]} \\
\mathbf{F}^{[KK]} &= -\epsilon_{ijk}\, dx^i \wedge dx^j \left[\exp[-2U] \left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right)\right] \\
\mathbf{F}^{\Lambda} &= \mathbb{C}^{\Lambda M}\partial_i Z_M\, dx^i \wedge (dt + \mathbf{A}^{[KK]}) \\
&\quad + \epsilon_{ijk} dx^i \wedge dx^j \left[\exp[-U] \left(\mathrm{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma} \left(\partial^k Z_\Sigma + \mathrm{Re}\mathcal{N}_{\Sigma\Gamma}\partial^k Z^\Gamma\right)\right]
\end{aligned}
$$

To any solution of the euclidian sigma model field equations we can associate a static solution of Supergravity and all static solutions are retrieved in this way.

The charges are defined by integration on all non trivial homology two-spheres

$$
\begin{aligned}
\mathcal{Q}_\alpha \equiv \begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix}_\alpha &= \frac{1}{4\pi\sqrt{2}} \int_{\mathbb{S}^2_\alpha} \epsilon_{ijk} dx^i \wedge dx^j \left[\exp[-U]\, \mathcal{M}_4\, \partial^k Z \right. \\
&\quad \left. + \exp[-2U] \left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right) \mathbb{C}\, Z\right]
\end{aligned}
$$

Already known, Bossard, Nicolai et al

# Two Main Results

I. A finite, four step algorithm that constructs the generic supergravity solution associated with every nilpotent orbit:

a) Nilpotent algebra associated with {h,X,Y}

b) Harmonic function parametrization of the coset representative in the symmetric gauge

c) Universal formula for the transition to the triangular gauge

d) Extraction of the supergravity fields from the triangular coset representative.

II. Reduction to the Tits Satake subalgebra and organization of supergravity models in a finite list of TS universality classes.

CRUCIAL INGREDIENT

**NEW**

# Nilpotent orbits & subalgebras

**The standard triple**

$$[h, X] = 2\,X \quad ; \quad [h, Y] = -2\,Y \quad ; \quad [X, Y] = 2\,h$$

**The nilpotent algebra**

$$\begin{aligned} [h\,,\,C_\mu] &= \mu\, C_\mu \\ \mathbb{N} &= \text{span}\,[C_2\,,\,C_3\,,\,\dots\,,\,C_{max}] \\ \mathbb{N}_{\mathbb{K}} &\equiv \mathbb{N} \bigcap \mathbb{K}^\star \quad ; \quad \mathbb{U} = \mathbb{H}^\star \oplus \mathbb{K}^\star \end{aligned}$$

**Graded decomposition**

$$\begin{aligned} \mathbb{N}_{\mathbb{K}} &= \bigoplus_{a=0}^{n} \mathbb{N}_{\mathbb{K}}^{(a)} \\ \mathcal{D}^i \mathbb{N}_{\mathbb{K}} &= \mathbb{N}_{K}^{(i)} \oplus \mathcal{D}^{i+1} \mathbb{N}_{\mathbb{K}} \end{aligned}$$

**Symmetric Coset representative**

$$\begin{aligned} \mathcal{Y}(x) &= \exp\,[\mathfrak{H}(\vec{x})] \\ \mathfrak{H}(\vec{x}) &= \sum_{\alpha=0}^{n} \underbrace{\sum_{i=1}^{\ell_\alpha} \mathfrak{h}_i^{(\alpha)}(\vec{x})\, A_\alpha^i}_{\in \mathbb{N}_{\mathbb{K}}^{(\alpha)}} \end{aligned}$$

# Hierarchical integration

$$
\begin{aligned}
\nabla^2 \mathfrak{h}_i^{(0)} &= 0 \\
\nabla^2 \mathfrak{h}_i^{(1)} &= \mathfrak{F}_i^{(1)}\left(\mathfrak{h}^{(0)}, \nabla \mathfrak{h}^{(0)}\right) \\
\nabla^2 \mathfrak{h}_i^{(2)} &= \mathfrak{F}_i^{(2)}\left(\mathfrak{h}^{(0)}, \nabla \mathfrak{h}^{(0)}, \mathfrak{h}^{(1)}, \nabla \mathfrak{h}^{(1)}\right) \\
\dots &= \dots \\
\nabla^2 \mathfrak{h}_i^{(n)} &= \mathfrak{F}_i^{(n)}\left(\mathfrak{h}^{(0)}, \nabla \mathfrak{h}^{(0)}, \mathfrak{h}^{(1)}, \nabla \mathfrak{h}^{(1)}, \dots, \mathfrak{h}^{(n-1)}, \nabla \mathfrak{h}^{(n-1)}\right),
\end{aligned}
$$

**This is the brilliant result of Bossard et al ………….**

**However from the symmmetric coset we cannot extract the supergravity fields**

**We need to transform the symmetric coset to the TRIANGULAR GAUGE!**

# General formula for the solvable gauge representative

$$\mathbb{L}(\mathcal{Y})\,\mathcal{Q}(\mathcal{Y}) = \mathcal{Y} \quad ; \quad \mathcal{Q}(\mathcal{Y}) \in \mathsf{H}^\star$$

$$\mathbb{L}(\mathcal{Y}) = \begin{pmatrix} L_{1,1}(\mathcal{Y}) & L_{1,2}(\mathcal{Y}) & \cdots & L_{1,n-1}(\mathcal{Y}) & L_{1,n}(\mathcal{Y}) \\ 0 & L_{2,2}(\mathcal{Y}) & \cdots & L_{2,n-1}(\mathcal{Y}) & L_{2,n}(\mathcal{Y}) \\ 0 & 0 & L_{3,3}(\mathcal{Y}) & \cdots & L_{3,n}(\mathcal{Y}) \\ \vdots & \dots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & L_{3,n}(\mathcal{Y}) \end{pmatrix}$$

**Determinants**

$$\mathfrak{D}_i(\mathcal{Y}) := \mathrm{Det}\begin{pmatrix} \mathcal{Y}_{1,1} & \cdots & \mathcal{Y}_{1,i} \\ \vdots & \vdots & \vdots \\ \mathcal{Y}_{i,1} & \cdots & \mathcal{Y}_{i,i} \end{pmatrix}, \quad \mathfrak{D}_0(\mathcal{Y}) := 1\,.$$

**Matrix elements**

$$\left(\mathbb{L}(\mathcal{Y})^{-1}\right)_{ij} \equiv \frac{1}{\sqrt{\mathfrak{D}_i(\mathcal{Y})\mathfrak{D}_{i-1}(\mathcal{Y})}}\mathrm{Det}\begin{pmatrix} \mathcal{Y}_{1,1} & \cdots & \mathcal{Y}_{1,i-1} & \mathcal{Y}_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{Y}_{i,1} & \cdots & \mathcal{Y}_{i,i-1} & \mathcal{Y}_{i,j} \end{pmatrix}$$

# Extraction of the supergravity fields

$$\mathbb{L}(\Phi) = \exp\left[-a\, L_{+}^{E}\right] \exp\left[\sqrt{2}\, Z^{M}\, \mathcal{W}_{M}\right] \mathbb{L}_{4}(\phi) \exp\left[U\, L_{0}^{E}\right]$$

**Universal decomposition of the D=3 Lie algebra**

$$\mathrm{adj}(\mathbb{U}_{D=3}) = \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_{E}) \oplus W_{(2,\mathbf{W})}$$

| Name of orbit | pq charges | Quart. Inv. $\mathfrak{I}_4$ | $\mathbf{W}$ – stab. group $\mathcal{S}_{\mathbf{W}} \subset \mathfrak{sl}(2,\mathbb{R})$ | $\mathrm{H}^\star$ – stab. group $\mathfrak{S}_{\mathrm{H}^\star} \subset \widehat{\mathfrak{sl}(2,\mathbb{R})} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star}$ | dim $\mathbb{N}$ | dim $\mathbb{N} \bigcap \mathbb{K}^\star$ |
|---|---|---|---|---|---|---|
| $\mathcal{O}^4_{11}$ | $\begin{pmatrix} 0 \\ 0 \\ 0 \\ q \end{pmatrix}$ | $0$ | $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ | $\underbrace{\mathrm{ISO}(1,1)}_{3 \text{ gen.}}$ | 3 | 3 |
| $\mathcal{O}^2_{11}$ | $\begin{pmatrix} \sqrt{3}\,p \\ 0 \\ 0 \\ 0 \end{pmatrix}$ | $0$ | $\mathbf{1}$ | $\underbrace{\mathrm{SO}(1,1) \triangleright \mathbb{R}}_{2 \text{ gen.}}$ | 4 | 3 |
| $\mathcal{O}^3_{11}$ | $\begin{pmatrix} 0 \\ p \\ -\sqrt{3}q \\ 0 \end{pmatrix}$ | $9\,p\,q^3 > 0$ | $\mathbb{Z}_3$ | $\underbrace{\mathbb{R}}_{1 \text{ gen. } A^2=0}$ | 5 | 4 |
| $\mathcal{O}^3_{22}$ | $\begin{pmatrix} 0 \\ p \\ \sqrt{3}q \\ 0 \end{pmatrix}$ | $-9\,p\,q^3 < 0$ | $\mathbf{1}$ | $\underbrace{\mathbb{R}}_{1 \text{ gen. } A^3=0}$ | 3 | 3 |
| $\mathcal{O}^1_{11}$ | $\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}p \\ 0 \\ \frac{7}{6}p \\ \sqrt{2}q \end{pmatrix}$ | $\frac{1}{128}p^3 \times (49p + 72q)$ | $\mathbf{1}$ | $\mathbf{1}$ | 6 | 4 |

Table 2: Properties of the $\mathfrak{g}_{(2,2)}$ orbits in the $S^3$ model. The structure of the electromagnetic charge vector is that obtained for solutions with vanishing Taub-NUT current. The symbol $\triangleright$ is meant to denote semidirect product. $\mathcal{S}_{\mathbf{W}}$ denotes the subgroup of the $D = 4$ duality group which leaves the charge vector invariant, while $\mathfrak{S}_{\mathrm{H}^\star}$ denotes the subgroup of the $\mathrm{H}^\star$ isotropy group of the $D = 3$ sigma-model which leaves invariant the $X$ element of the standard triple. This latter is the Lax operator in the one-dimensional spherical symmetric approach.

A generic element of the corresponding Lie algebra is a linear combination of three generators $J, T_1, T_2$, satisfying the commutation relations:

$$[J\,,\,T_1] \;=\; \frac{1}{\sqrt{2}}\,T_1 + \frac{3}{2\sqrt{6}}\,T_2$$

# Example of solution in the non BPS orbit o322

$$\exp[-U] = \sqrt{\mathcal{H}_2\mathcal{H}_3^3 - 4a_1^2}$$
$$\mathrm{Im}\, z = \frac{\sqrt{\mathcal{H}_2\mathcal{H}_3^3 - 4a_1^2}}{\mathcal{H}_3^2}$$
$$\mathrm{Re}\, z = -\frac{2a_1}{\mathcal{H}_3^2}$$

$$\mathbf{A}^{[KK]} = 0$$
$$j^{EM} = \star\nabla\begin{pmatrix} 0 \\ -\frac{\mathcal{H}_2}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}}\mathcal{H}_3 \\ 0 \end{pmatrix}$$

$$\nabla^2\mathcal{H}_{1,2} = 0$$

**All the fields are parametrerized in terms of two harmonic functions**
**Assigning the position of the**
**poles we have multicenter non spherical symmetric solutions**

# Structure of the algebras and Tits Satake projection

The non compact rank of a non compact coset is defined as

$$r_{nc} = \mathrm{rank}\,(\mathrm{U}/\mathrm{H}) \equiv \dim \mathcal{H}^{n.c.}$$
$$\mathcal{H}^{n.c.} \equiv \mathrm{CSA}_{\mathbb{U}(\mathbb{C})} \bigcap \mathbb{K}$$

**If the Cartan subalgebra contains also compact elements U/H is non maximally split**

Tits Satake is a geometrical projection of the Root system on its non compact subspace

$$\Pi^{TS} \quad ; \quad \Delta_{\mathbb{U}} \mapsto \overline{\Delta}_{\mathbb{U}^{TS}}$$

Concept of Paint Group (P.F. VanProeyen, Rulik, Trigiante 2007)

$$\mathbb{G}_{\mathrm{paint}} = \mathrm{Aut}_{\mathrm{Ext}}\left[Solv_{\mathrm{U/H}}\right] = \frac{\mathrm{Aut}\left[Solv_{\mathrm{U/H}}\right]}{Solv_{\mathrm{U/H}}}$$

The Tits Satke projection commutes with the dimensional reduction

$$\begin{array}{rcl}\mathrm{adj}(\mathbb{U}_{D=3}) & = & \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E) \oplus W_{(2,W)} \\ & \Downarrow & \\ \mathrm{adj}(\mathbb{U}^{TS}_{D=3}) & = & \mathrm{adj}(\mathbb{U}^{TS}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E) \oplus W_{(2,W^{TS})}\end{array}$$

# Reduction to TS

- A careful group theoretical analysis reveals that in the adjoint rep. of $U_{D=3}$ we have always enough parameters to reduce charges and fields to the Tits Satake subalgebra.
- Essentially we can delete entire representations of the subpaint group.

$$\mathfrak{t} \in \mathbb{G}_{\text{subpaint}} \subset \mathbb{G}_{\text{paint}} \subset \mathbb{U} \quad \Leftrightarrow \quad [\mathfrak{t}, \mathbb{G}_{\text{TS}}] = 0$$

| # | TS D=4 | TS D=3 | coset D=4 | coset D=3 | Paint Group | subP Group | susy |
|---|---|---|---|---|---|---|---|
| 1 | $\frac{E_{7(7)}}{SU(8)}$ | $\frac{E_{8(8)}}{SO^\star(16)}$ | $\frac{E_{7(7)}}{SU(8)}$ | $\frac{E_{8(8)}}{SO^\star(16)}$ | 1 | 1 | $\mathcal{N}=8$ |
| 2 | $\frac{SU(1,1)}{U(1)}$ | $\frac{G_{2(2)}}{SL(2,R)\times SL(2,R)}$ | $\frac{SU(1,1)}{U(1)}$ | $\frac{G_{2(2)}}{SL(2,R)\times SL(2,R)}$ | 1 | 1 | $\mathcal{N}=2$ n=1 |
| 3 | $\frac{Sp(6,R)}{SU(3)\times U(1)}$ | $\frac{F_{4(4)}}{Sp(6,R)\times SL(2,R)}$ | $\frac{Sp(6,R)}{SU(3)\times U(1)}$ | $\frac{F_{4(4)}}{Sp(6,R)\times SL(2,R)}$ | 1 | 1 | $\mathcal{N}=2$ $n=6$ |
| 4 | | | $\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$ | $\frac{E_{6(2)}}{SU(3,3)\times SL(2,R)}$ | $SO(2)\times SO(2)$ | 1 | $\mathcal{N}=2$ $n=9$ |
| 5 | | | $\frac{SO^\star(12)}{SU(6)\times U(1)}$ | $\frac{E_{7(-5)}}{SO^\star(12)\times SL(2,R)}$ | $SO(3)\times SO(3)$ $\times SO(3)$ | $SO(3)_d$ | $\mathcal{N}=6$ $\mathcal{N}=2$ n=15 |
| 6 | | | $\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$ | $\frac{E_{8(-24)}}{E_{7(-25)}\times SL(2,R)}$ | SO(8) | $G_{2(2)}$ | $\mathcal{N}=2$ $n=27$ |
| 7 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,1)}{SO(2)}$ | $\frac{SO(4,3)}{SO(2,2)\times SO(2,1)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,1)}{SO(6)}$ | $\frac{SO(8,3)}{SO(6,2)\times SO(2,1)}$ | SO(5) | SO(4) | $\mathcal{N}=4$ n=1 |
| 8 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(3,2)}{SO(3)\times SO(2)}$ | $\frac{SO(5,4)}{SO(3,2)\times SO(2,2)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,2)}{SO(6)\times SO(2)}$ | $\frac{SO(8,4)}{SO(6,2)\times SO(2,2)}$ | SO(4) | SO(3) | $\mathcal{N}=4$ n=2 |
| 9 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(4,3)}{SO(4)\times SO(3)}$ | $\frac{SO(6,5)}{SO(4,2)\times SO(2,3)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,3)}{SO(6)\times SO(3)}$ | $\frac{SO(8,5)}{SO(6,2)\times SO(2,3)}$ | SO(3) | SO(2) | $\mathcal{N}=4$ n=3 |
| 10 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(5,4)}{SO(5)\times SO(4)}$ | $\frac{SO(7,6)}{SO(5,2)\times SO(2,4)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,4)}{SO(6)\times SO(4)}$ | $\frac{SO(8,6)}{SO(6,2)\times SO(2,4)}$ | SO(2) | 1 | $\mathcal{N}=4$ n=4 |
| 11 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,5)}{SO(6)\times SO(5)}$ | $\frac{SO(8,7)}{SO(6,2)\times SO(2,5)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,5)}{SO(6)\times SO(5)}$ | $\frac{SO(8,7)}{SO(6,2)\times SO(2,5)}$ | 1 | 1 | $\mathcal{N}=4$ n=5 |
| 12 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,6)}{SO(6)\times SO(6))}$ | $\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,6)}{SO(6)\times SO(6))}$ | $\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$ | 1 | 1 | $\mathcal{N}=4$ n=6 |
| 13 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,7)}{SO(6)\times SO(7))}$ | $\frac{SO(8,9)}{SO(6,2)\times SO(2,7)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(6,6+p)}{SO(6)\times SO(6+p))}$ | $\frac{SO(8,8+p)}{SO(6,2)\times SO(2,6+p)}$ | SO(p) | $SO(p-1)$ | $\mathcal{N}=4$ n=6+p |
| 14 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,1)}{SO(2)}$ | $\frac{SO(4,3)}{SO(2,2)\times SO(2,1)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,1)}{SO(2)}$ | $\frac{SO(4,3)}{SO(2,2)\times SO(2,1)}$ | 1 | 1 | $\mathcal{N}=2$ n=2 |
| 15 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,2)}{SO(2)\times SO(2)}$ | $\frac{SO(4,4)}{SO(2,2)\times SO(2,2)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,2)}{SO(2)\times SO(2)}$ | $\frac{SO(4,4)}{SO(2,2)\times SO(2,2)}$ | 1 | 1 | $\mathcal{N}=2$ n=3 |
| 16 | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,3)}{SO(2)\times SO(3)}$ | $\frac{SO(4,5)}{SO(2,2)\times SO(2,3)}$ | $\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,2+p)}{SO(2)\times SO(2+p)}$ | $\frac{SO(4,4+p)}{SO(2,2)\times SO(2,2+p)}$ | SO(p) | $SO(p-1)$ | $\mathcal{N}=2$ n=3+p |

Table 3: The 16 instances of *non-exotic* homogenous symmetric scalar manifolds appearing in $D=4$ supergravity. Non exotic means that the Tits Satake projection of the root system is a standard Lie Algebra root system. The 16 models are grouped according to their Tits Satake Universality classes. The time-like dimensional reduction is listed side by side. Within each class the models are distinguished by the different structure of the Paint Group and of its subPaint subgroup. The Paint group is the same in D=4 and in D=3

| # | TS D=4 | TS D=3 | coset D=4 | coset D=3 | Paint Group | subP Group | susy |
|---|---|---|---|---|---|---|---|
| $1_e$ | $bc_1$ | $bc_2$ | $\frac{\mathrm{SU(p+1,1)}}{\mathrm{SU(p+1)\times U(1)}}$ | $\frac{\mathrm{SU(p+2,2)}}{\mathrm{SU(p+1,1)\times SL(2,R)_{h^\star}}}$ | $\mathrm{U(1)\times U(1)\times U(p)}$ | $\mathrm{U(p-1)}$ | $\mathcal{N}=2$ n=p+1 |
| $2_e$ | $bc_3$ | $bc_4$ | $\frac{\mathrm{SU(p+1,3)}}{\mathrm{SU(p+1)\times SU(3)\times U(1)}}$ | $\frac{\mathrm{SU(p+2,4)}}{\mathrm{SU(p+1,2)\times SU(1,2)\times U(1)}}$ | $\mathrm{U(1)\times U(1)\times U(p)}$ | $\mathrm{U(p-1)}$ | $\mathcal{N}=3$ n=p+1 |
| $3_e$ | $bc_1$ | $bc_2$ | $\frac{\mathrm{SU(5,1)}}{\mathrm{SU(5)\times U(1)}}$ | $\frac{\mathrm{E_{6(-14)}}}{\mathrm{SO^\star(10)\times SO(2)}}$ | $\mathrm{U(1)\times U(1)\times U(4)}$ | $\mathrm{U(3)}$ | $\mathcal{N}=5$ |

Table 4: The 3 instances of *exotic* homogenous symmetric scalar manifolds appearing in $D=4$ supergravity. Exotic means that the Tits Satake projection of the root system is not a standard Lie Algebra root system. Notwithstanding this anomaly the concept of Paint Group, according to its definition as group of external automorphisms of the solvable Lie algebra generating the non compact coset manifold still exists. The Paint group is the same in D=4 and in D=3

### 6.3 Dynkin diagram analysis of the principal models

Next we analyze the form of the root systems of the $\mathbb{U}_{\mathrm{D=3}}$ algebras in relation with the decomposition (2.20).

**$\mathcal{N}$=8** This is the case of maximal supersymmetry and it is illustrated by fig. 1.

In this case all the involved Lie algebras are maximally split and we have

$$\mathrm{adj}\,\mathrm{E}_{8(8)} = \mathrm{adj}\,\mathrm{E}_{7(7)} \oplus \mathrm{adj}\,\mathrm{SL}(2,\mathbb{R})_{\mathrm{E}} \oplus (\mathbf{2},\mathbf{56}) \tag{6.8}$$

The highest root of $\mathrm{E}_{8(8)}$ is

$$\psi = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7 + 2\alpha_8 \tag{6.9}$$

and the unique simple root not orthogonal to $\psi$ is $\alpha_8 = \alpha_W$, according to the labeling of roots as in fig. 1. This root is the highest weight of the fundamental **56**-representation of $E_{7(7)}$.

| # | $\mathbb{U}_{D=3}$ | $\mathbb{H}^\star$ | $\mathbb{K}^\star$ | $\mathbb{U}_{D=4}$ | rep.$W$ | $\mathbb{H}_c$ |
|---|---|---|---|---|---|---|
| 1 | $\mathfrak{e}_{8(8)}$ | $\mathfrak{so}^\star(16)$ | $\mathbf{128}_s$ | $\mathfrak{e}_{7(7)}$ | $\mathbf{56}$ | $\mathfrak{su}(8)$ |
| 2 | $\mathfrak{g}_{2(2)}$ | $\widehat{\mathfrak{sl}(2,\mathrm{R})} \oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $(\mathbf{4}_{3/2}\,,\,\mathbf{2}_{h^\star})$ | $\mathfrak{sl}(2,\mathrm{R})$ | $\mathbf{4}_{3/2}$ | $\mathfrak{so}(2)$ |
| 3 | $\mathfrak{f}_{4(4)}$ | $\widehat{\mathfrak{sp}(6,\mathrm{R})} \oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{14}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{sp}(6,\mathrm{R})$ | $\mathbf{14}$ | $\mathfrak{u}(3)$ |
| 4 | $\mathfrak{e}_{6(2)}$ | $\widehat{\mathfrak{su}(3,3)} \oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{20}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{su}(3,3)$ | $\mathbf{20}$ | $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ $\oplus \mathfrak{u}(1)$ |
| 5 | $\mathfrak{e}_{7(-5)}$ | $\widehat{\mathfrak{so}^\star(12)} \oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{32}_{spin}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{so}^\star(12)$ | $\mathbf{32}_{spin}$ | $\mathfrak{u}(6)$ |
| 6 | $\mathfrak{e}_{8(-24)}$ | $\widehat{\mathfrak{e}_{7(-25)}} \oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{56}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{e}_{7(-25)}$ | $\mathbf{56}$ | $\mathfrak{u}(6)$ |
| 7 | $\mathfrak{so}(8,3)$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,1)$ | $(\mathbf{8}\,,\,\mathbf{3})$ | $\mathfrak{so}(6,1) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{7},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{u}(1)$ |
| 8 | $\mathfrak{so}(8,4)$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,2)$ | $(\mathbf{8}\,,\,\mathbf{4})$ | $\mathfrak{so}(6,2) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{8},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{so}(2)$ $\oplus \mathfrak{u}(1)$ |
| 9 | $\mathfrak{so}(8,5)$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,3)$ | $(\mathbf{8}\,,\,\mathbf{5})$ | $\mathfrak{so}(6,3) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{9},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{so}(3)$ $\oplus \mathfrak{u}(1)$ |
| 10 | $\mathfrak{so}(8,6)$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,4)$ | $(\mathbf{8}\,,\,\mathbf{6})$ | $\mathfrak{so}(6,4) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{10},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{so}(4)$ $\oplus \mathfrak{u}(1)$ |
| 11 | $\mathfrak{so}(8,7)$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,5)$ | $(\mathbf{8}\,,\,\mathbf{7})$ | $\mathfrak{so}(6,5) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{11},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{so}(5)$ $\oplus \mathfrak{u}(1)$ |
| 12 | $\mathfrak{so}(8,8)$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,6)$ | $(\mathbf{8}\,,\,\mathbf{8})$ | $\mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{12},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{so}(6)$ $\oplus \mathfrak{u}(1)$ |
| 13 | $\mathfrak{so}(8,8+\mathrm{p})$ | $\mathfrak{so}(6,2) \oplus \mathfrak{so}(2,6+p)$ | $(\mathbf{8}\,,\,\mathbf{8+p})$ | $\mathfrak{so}(6,6+p) \oplus \mathfrak{sl}(2,\mathrm{R})$ | $(\mathbf{12+p},\mathbf{2})$ | $\mathfrak{so}(6) \oplus \mathfrak{so}(6+p)$ $\oplus \mathfrak{u}(1)$ |
| 14 | $\mathfrak{so}(4,3)$ | $\widehat{\mathfrak{sl}(2,\mathrm{R})} \oplus \widehat{\mathfrak{so}(2,1)}$ $\oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{2}}\,,\,\widehat{\mathbf{3}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{sl}(2,\mathrm{R}) \oplus \mathfrak{so}(2,1)$ | $(\mathbf{2}\,,\,\mathbf{3})$ | $\mathfrak{so}(2) \oplus \mathfrak{u}(1)$ |
| 15 | $\mathfrak{so}(4,4)$ | $\widehat{\mathfrak{sl}(2,\mathrm{R})} \oplus \widehat{\mathfrak{so}(2,2)}$ $\oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{2}}\,,\,\widehat{\mathbf{4}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{sl}(2,\mathrm{R}) \oplus \mathfrak{so}(2,2)$ | $(\mathbf{2}\,,\,\mathbf{4})$ | $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ $\oplus \mathfrak{u}(1)$ |
| 16 | $\mathfrak{so}(4,4+\mathrm{p})$ | $\widehat{\mathfrak{sl}(2,\mathrm{R})} \oplus \widehat{\mathfrak{so}(2,2+p)}$ $\oplus \mathfrak{sl}(2,\mathrm{R})_{h^\star}$ | $\left(\widehat{\mathbf{2}}\,,\,\widehat{\mathbf{4+p}}\,,\,\mathbf{2}_{h^\star}\right)$ | $\mathfrak{sl}(2,\mathrm{R}) \oplus \mathfrak{so}(2,2)$ | $(\mathbf{2}\,,\,\mathbf{4+p})$ | $\mathfrak{so}(2) \oplus \mathfrak{so}(2+p)$ $\oplus \mathfrak{u}(1)$ |

Table 6: Table of $\mathbb{H}^\star$ subalgebras of $\mathbb{U}_{D=3}$, $\mathbb{K}^\star$-representations and $\mathbf{W}$ representations of $\mathbb{U}_{D=4}$ for the supergravity models based on *non-exotic* scalar symmetric spaces

What is important to stress is that, although isomorphic $\mathbb{H}^\star$ and $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{U}_{D=4}$ are different subalgebras of $\mathbb{U}_{D=3}$:

$$\mathbb{U}_{D=3} \supset \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star} \neq \mathfrak{sl}(2,\mathbb{R})_E \subset \mathbb{U}_{D=3} \quad ; \quad \mathbb{U}_{D=3} \supset \widehat{\mathbb{U}_{D=4}} \neq \mathbb{U}_{D=4} \subset \mathbb{U}_{D=3} \tag{8.5}$$

Morevover, while the decomposition (2.20) is universal and holds true for all supergravity models, the structure (8.3) of the $\mathbb{H}^\star$ subalgebra is peculiar to the $\mathcal{N}=2$ models. In other cases the structure of $\mathbb{H}^\star$ is different.

The reduction to the Tits Satake projection however is universal and applies to all non maximally split cases.

Indeed the remaining cases are of the form:

$$\frac{\mathrm{U}_{D=3}}{\mathrm{H}^\star} = \frac{\mathrm{SO}(2+\mathrm{q},\mathrm{q}+2+\mathrm{p})}{\mathrm{SO}(\mathrm{q},2)\times\mathrm{SO}(2,\mathrm{q}+\mathrm{p})} \tag{8.6}$$