Max Planck Institute for Mathematics
California Institute of Technology

Geometry and Quantization

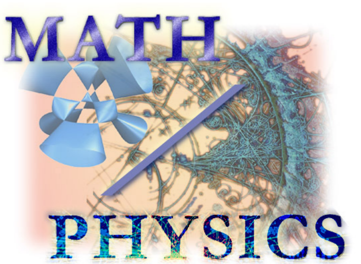
Sergei Gukov

based on: S.G., E.Witten, arXiv:0809.0305

S.G., T.Dimofte, arXiv:1003.4808

S.G., arXiv:1011.2218

"What are you talking about? — how can you have *half* a quantum theory?"




nan·o·sec·ond
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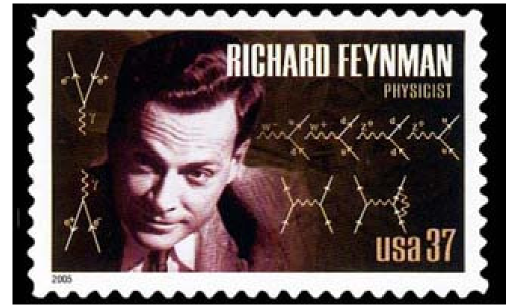
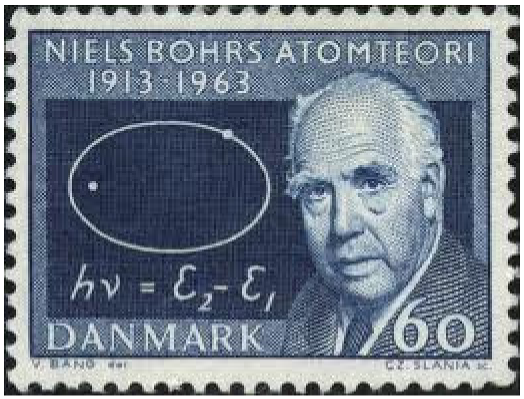
Quantum Mechanics

I think I can safely say that nobody understands quantum mechanics.
Richard Feynman

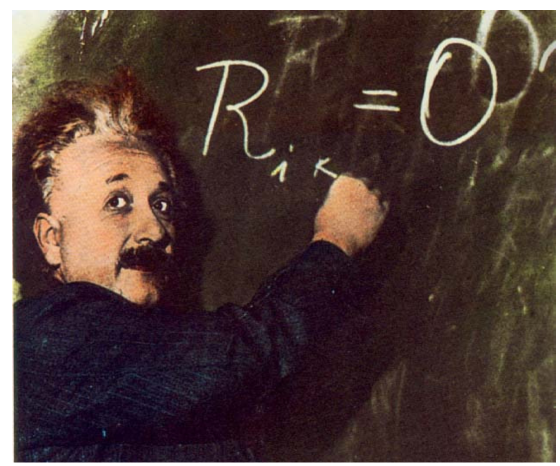




Anyone who is not shocked by quantum theory has not understood a single word.
Niels Bohr

Very interesting theory -- it makes no sense at all.
Groucho Marx

Gott würfelt nicht!
Albert Einstein



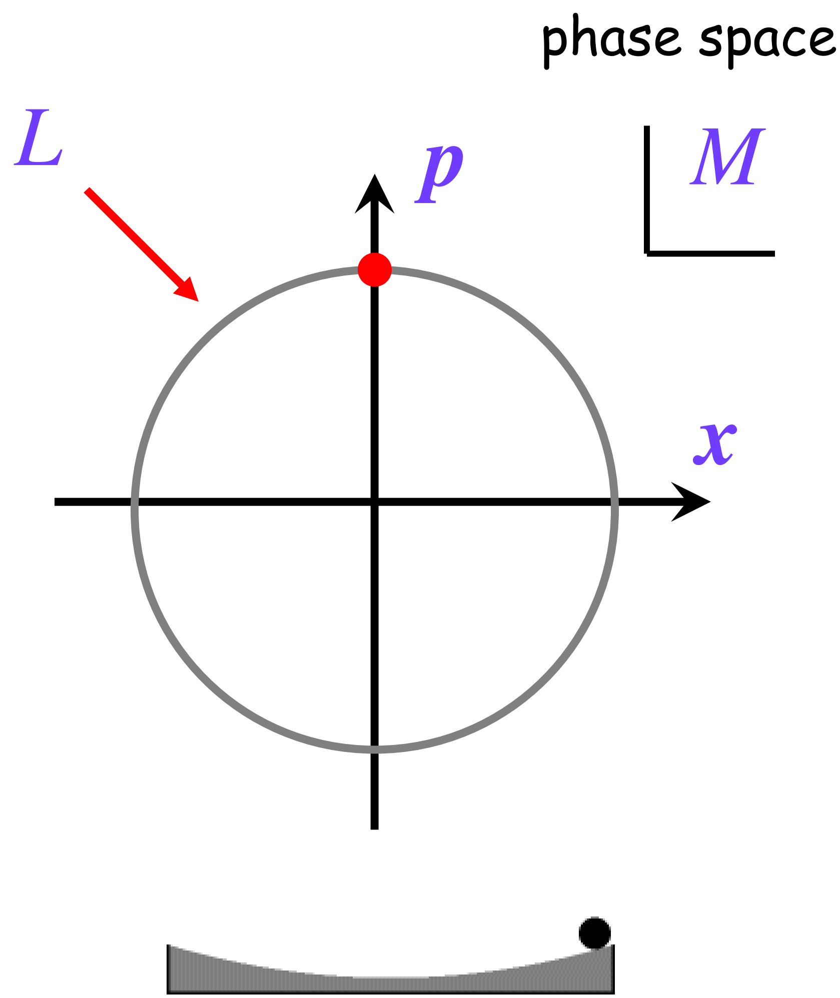
The more success the quantum theory has the sillier it looks.

Bohr-Sommerfeld quantization

$$H = \frac{1}{2}x^2 + \frac{1}{2}p^2$$


Niels Bohr

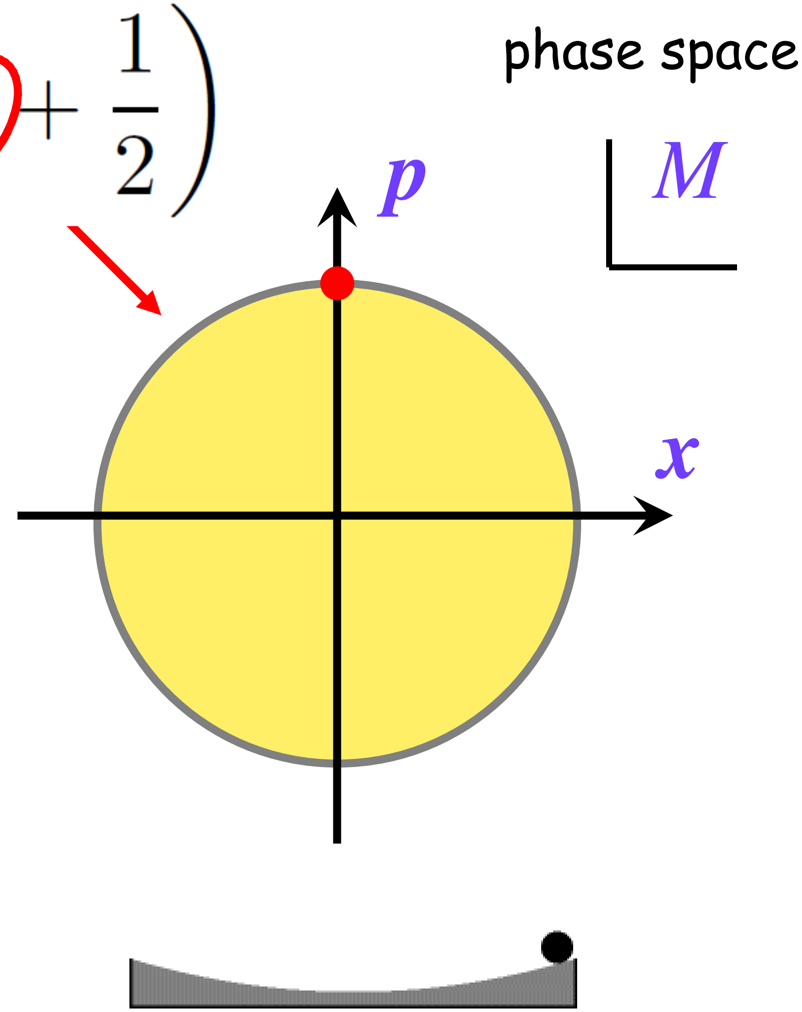

Arnold Sommerfeld



Bohr-Sommerfeld quantization

$$E = \frac{1}{2\pi}\int dp \wedge dx = \hbar\left(n + \frac{1}{2}\right)$$

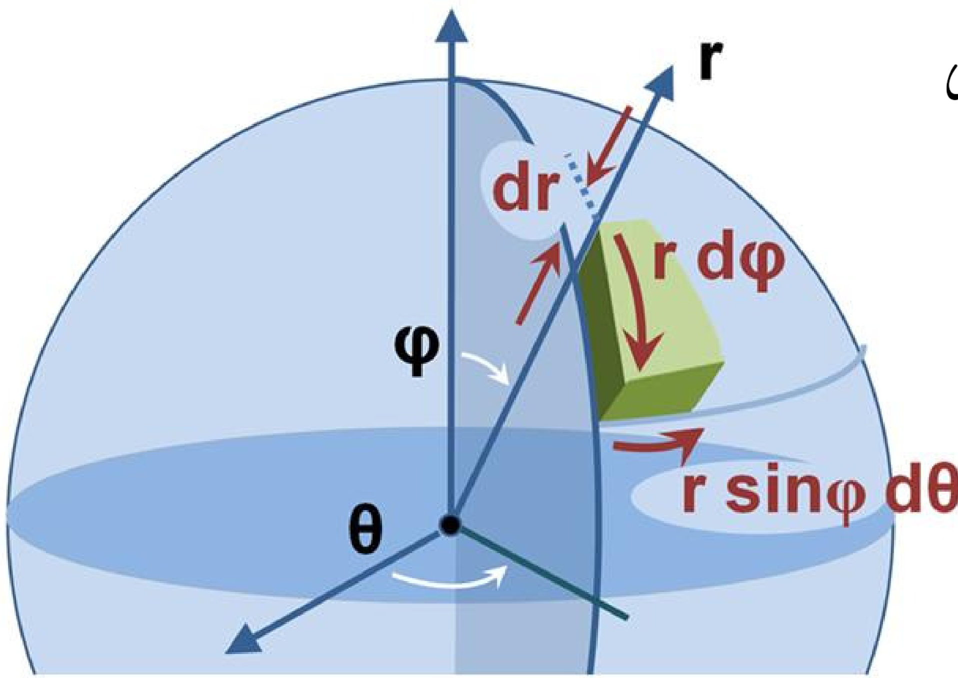
integer



Niels Bohr

Arnold Sommerfeld

Quantization of $M = S^2$



$$\omega = \frac{1}{4\pi\hbar} \sin\varphi \, d\varphi \wedge d\theta$$

$$\begin{cases} x &= r\sin\varphi\cos\theta \\ y &= r\sin\varphi\sin\theta \\ z &= r\cos\varphi \end{cases}$$

$x^2 + y^2 + z^2 = 1$

$$\Rightarrow \quad \omega = \frac{1}{4\pi\hbar} \frac{dx \wedge dy}{z}$$

$$\boxed{\dim \mathcal{H} = \int_M \omega = \frac{1}{\hbar}}$$

$$\hbar^{-1} \in \mathbb{Z}$$

"Quantum Symplectic Geometry"

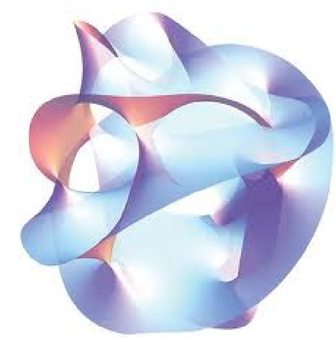
symplectic manifold (M, ω) $\rightsquigarrow$ $\mathcal{H}$ (Hilbert space)

algebra of functions on M $\rightsquigarrow$ $\mathcal{A}$ (algebra of operators)

Lagrangian submanifolds $L \subset M$ $\rightsquigarrow$ vectors $\psi \in \mathcal{H}$

Mirror Symmetry

<u>A-model:</u> symplectic manifold Y ⟷ <u>B-model:</u> complex manifold $\tilde{Y}$



Mirror Symmetry

<u>A-model:</u>

symplectic manifold

Y

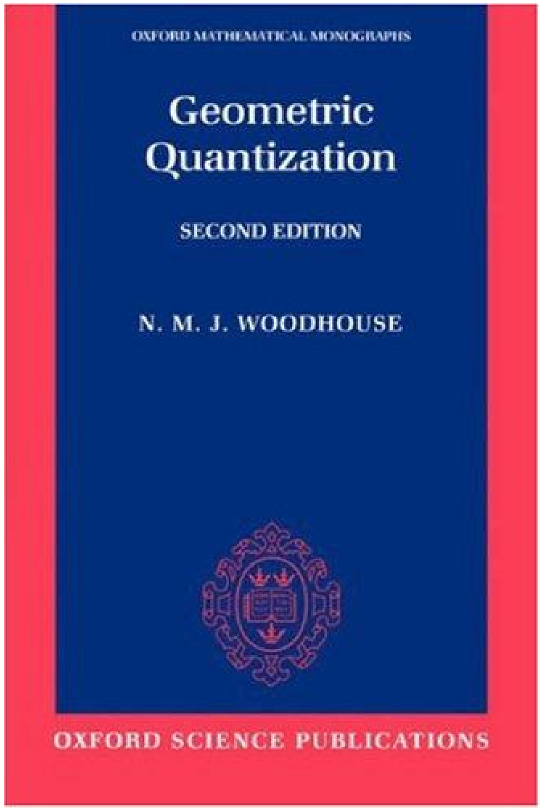
- Gromov-Witten invariants
- Fukaya category
- Quantum cohomology

...

Geometric Quantization

- $\mathcal{L} \to M$ "prequantum line bundle" with unitary connection of curvature $\boldsymbol{\omega}$

$$[\omega] \in H^2(M;\mathbb{Z})$$



- choice of polarization

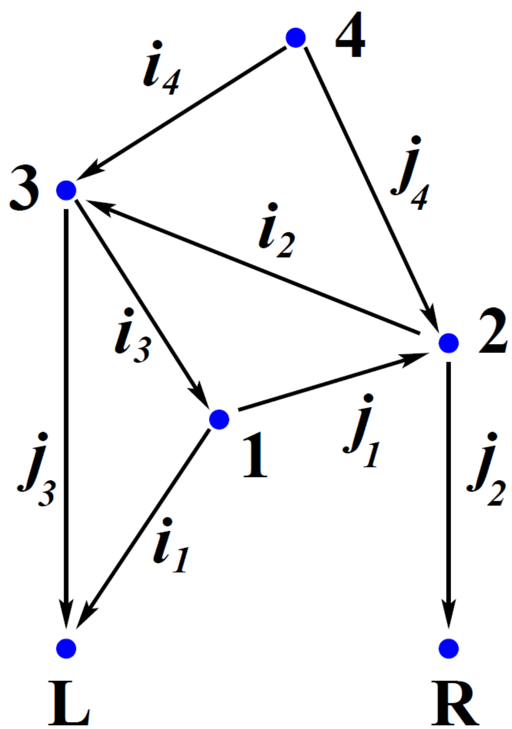
$$M \simeq T^*U$$

$$\ldots \rightsquigarrow \mathcal{H}, \mathcal{A}$$

Deformation Quantization

[F.Bayen, M.Flato, C.Fronsdal, A.Lichnerowicz, D.Sternheimer '78]
[M.Kontsevich '97]

$$f \star_\hbar g := \sum_{n=0}^{\infty} \hbar^n \sum_{\substack{\text{graphs } \Gamma \\ \text{of order } n}} w(\Gamma) B_\Gamma(f,g)$$



- no auxiliary choices, but:
  - no Hilbert space $\mathcal{H}$
  - formal deformation of the ring of functions on M

Deformation Quantization

Example: $M = S^2$ $\qquad \omega = \frac{1}{4\pi\hbar}\frac{dx \wedge dy}{z}$

$x, y, z \rightsquigarrow \hat{x}, \hat{y}, \hat{z}$

$[\hat{x}, \hat{y}] = \hbar \hat{z}$, *etc.*

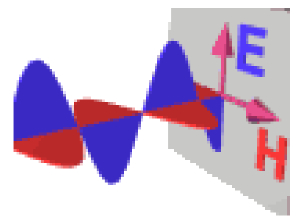
Lie algebra sl(2)

$x^2 + y^2 + z^2 = 1 \rightsquigarrow \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = 1$

Examples ...

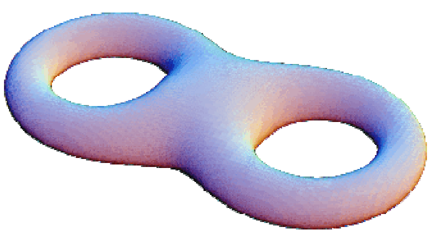
G = (simple) compact Lie group e.g. **SU(2)**

$\mathcal{A}$ = connection on a G-bundle $E \rightarrow C$ over a genus-g Riemann surface C



$M = \mathcal{M}_{\text{flat}}(G, C)$: space of solutions

$$dA + A \wedge A = 0$$

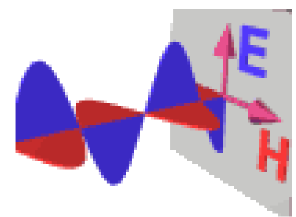


an example of a symplectic manifold!

Examples from Gauge Theory

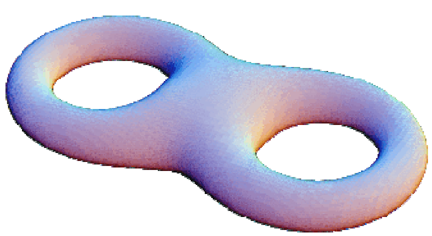
A flat connection on C is determined by its holonomies, that is by a homomorphism

$$\pi_1(C) \to G$$



$$M \;=\; \mathcal{M}_{\mathrm{flat}}(G,C)$$: space of solutions

$$dA + A \wedge A \;=\; 0$$



an example of a symplectic manifold!

Examples from Gauge Theory

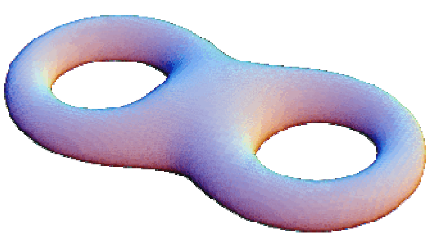
A flat connection on C is determined by its holonomies, that is by a homomorphism

$$\pi_1(C) \to G$$

more concretely,

$$A_1 B_1 A_1^{-1} B_1^{-1} \ \ldots \ A_g B_g A_g^{-1} B_g^{-1} = \mathbf{1}$$

holonomies

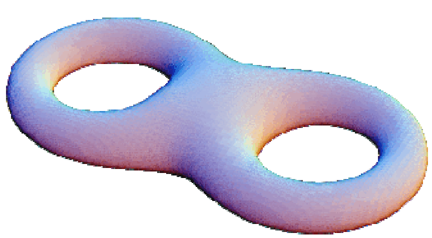


Examples from Gauge Theory

In total, the group elements $A_i, B_j, i, j = 1, \ldots, g$ contain 2g dim $\mathbf{G}$ real parameters, so that generically, for g > 1, after imposing the equation

$$A_1 B_1 A_1^{-1} B_1^{-1} \ \ldots \ A_g B_g A_g^{-1} B_g^{-1} = \mathbf{1}$$

and dividing by conjugation we obtain a space of real dimension



$$\boxed{\dim M = 2\,(g-1) \dim \mathbf{G}}$$

… compact, smooth*

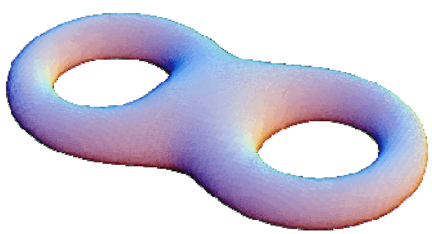
Examples from Gauge Theory

<u>Example:</u> $\mathbf{G} = \mathbf{SU(2)}$, g = 2

$$M \cong \mathbb{C}\mathbf{P}^3$$

nice symplectic manifold!

Note: $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$



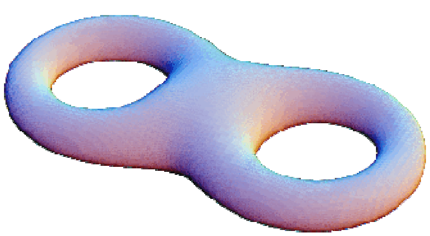
$$\boxed{\text{dim } M = 2\,(\text{g-1})\text{ dim } \mathbf{G}}$$

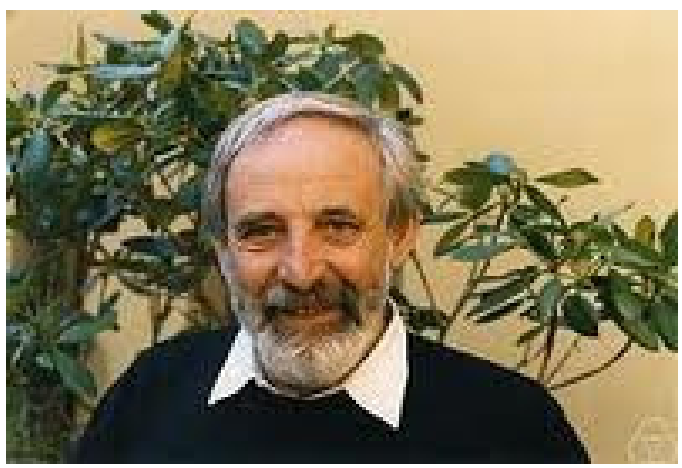
... compact, smooth*

Examples from Gauge Theory

The space $M = \mathcal{M}_{\mathrm{flat}}(G, C)$ comes equipped with a natural symplectic form:

$$\omega = \frac{1}{4\pi^2\hbar}\int_C \mathrm{Tr}\,\delta A \wedge \delta A$$





Raoul Bott

Sir Michael Atiyah

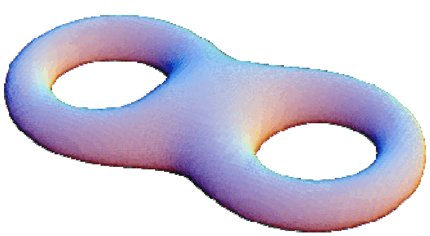
Examples from Gauge Theory

The space $M = \mathcal{M}_{\text{flat}}(G, C)$ comes equipped with a natural symplectic form:

$$\omega = \frac{1}{4\pi^2\hbar}\int_C \mathrm{Tr}\,\delta A\wedge\delta A$$

What is the corresponding Hilbert space $\mathcal{H}$?

Moreover, $H^2(M,\mathbb{Z})\cong\mathbb{Z}$



M is "quantizable" only for integer values of the level

$$k = \frac{1}{\hbar} \in \mathbb{Z}$$

Examples from Gauge Theory

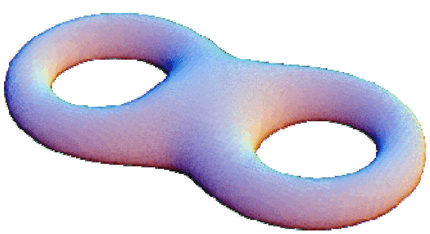
The space $M = \mathcal{M}_{\text{flat}}(G, C)$ is compact

$\mathcal{H}$ is finite-dimensional, and dim $\mathcal{H}$ is a polynomial in k, whose leading coefficient equals the volume of M:

$$\dim \mathcal{H} = \int_M \frac{\omega^n}{n!} + \ldots$$

<u>Example:</u> g=2

G=SU(2)



$$\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{C}\mathbf{P}^3) = \frac{1}{6} k^3$$

Verlinde Formula

The space $M = \mathcal{M}_{\text{flat}}(G, C)$ is compact

$\mathcal{H}$ is finite-dimensional, and dim $\mathcal{H}$ is a polynomial in *k*:

$$\dim \mathcal{H} = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g}$$

E.Verlinde

G=SU(2)

<u>Example:</u> g=2

$$\dim \mathcal{H} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1$$

Verlinde Formula

In general, the Verlinde formula has the following form:

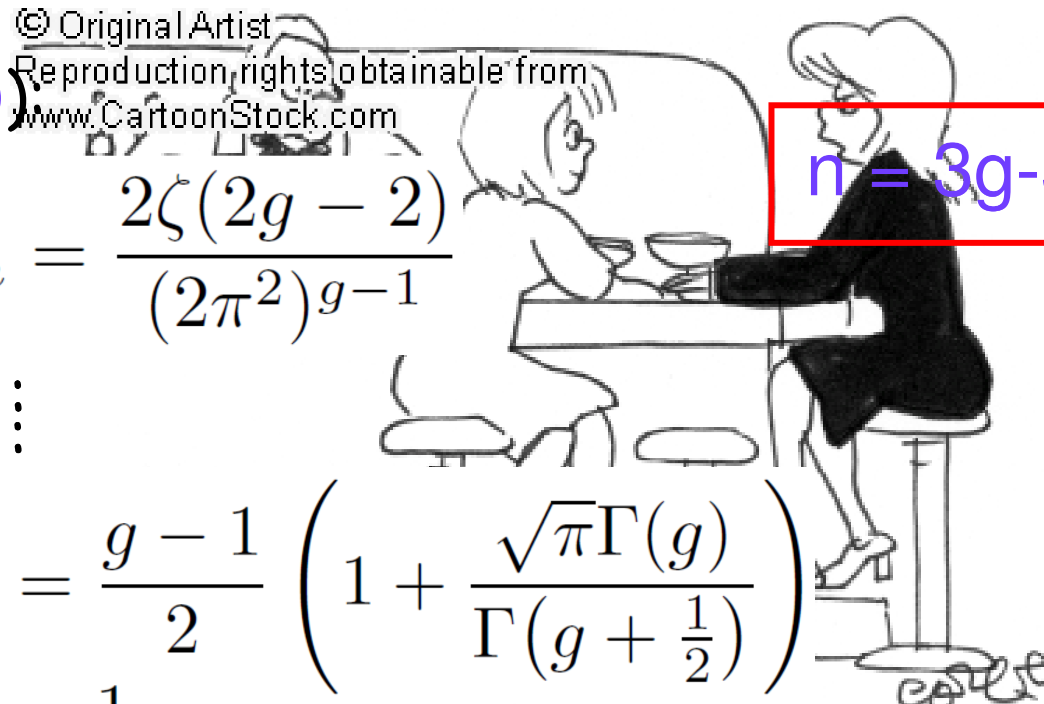
$$\dim \mathcal{H} \; = \; a_n k^n + a_{n-1} k^{n-1} + \ldots + a_1 k + a_0$$

where (for **G=SU(2)**):

$$\boxed{n = 3g-3}$$

$$\begin{cases} a_n = \dfrac{2\zeta(2g-2)}{(2\pi^2)^{g-1}} \\ \vdots \\ a_1 = \dfrac{g-1}{2}\left(1 + \dfrac{\sqrt{\pi}\,\Gamma(g)}{\Gamma(g+\frac{1}{2})}\right) \\ a_0 = 1 \end{cases}$$

E.Verlinde



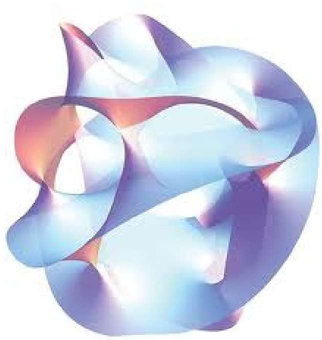
"I dated him to get my car repaired,
but he turned out to be a quantum mechanic!"

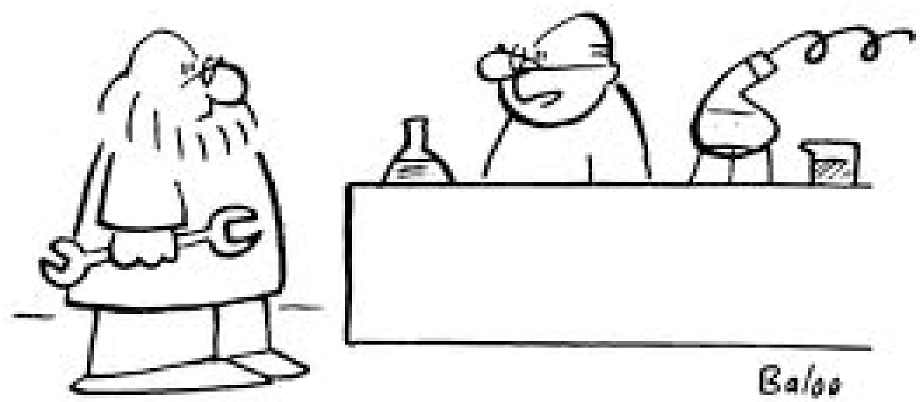
Verlinde Formula

In general, the Verlinde formula has the following form:

$$\dim \mathcal{H} \;=\; \underbrace{a_n k^n + a_{n-1}}_{} k^{n-1} + \underbrace{\ldots + a_1 k + a_0}_{}$$

- Modern approach to quantization offers an interpretation of the coefficients a_i via classical geometry of moduli spaces.





"Blumenkraft, I'm afraid you have the wrong idea about quantum mechanics."

Verlinde Formula

In general, the Verlinde formula has the following form:

$$\dim \mathcal{H} = \underbrace{a_n k^n + a_{n-1} k^{n-1}} + \ldots + \underbrace{a_1 k + a_0}$$

$$\hbar = \frac{1}{k} \to 0 \quad \xleftrightarrow{\text{mirror symmetry}} \quad {}^L\hbar = -\frac{1}{\hbar} \to 0$$

have a simple interpretation in terms of classical geometry of Y, the moduli space associated with the structure group $\mathbf{G}$

classical geometry of $\widetilde{\mathrm{Y}}$, the moduli space for a different group ${}^L\mathbf{G}$

Langlands duality

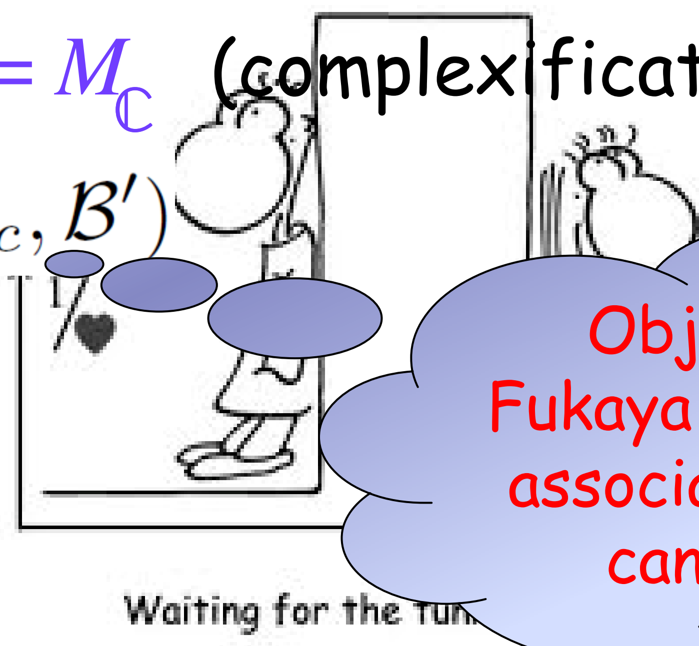
Galois representations of $\mathbf{G}$		automorphic representations of ${}^L\mathbf{G}$
$\mathbf{U(N)}$	$\longleftrightarrow$	$\mathbf{U(N)}$
$\mathbf{SO(2N)}$		$\mathbf{SO(2N)}$
$\mathbf{SO(2N+1)}$		$\mathbf{Sp(2N)}$
$\mathbf{E6}$		$\mathbf{E6/Z_3}$
$\mathbf{E8}$		$\mathbf{E8}$

Robert Langlands

Brane Quantization

A-model of $Y = M_{\mathbb{C}}$ (complexification of M)

$$\mathcal{H} \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$$



Objects in the Fukaya category of ***Y*** associated to ***M*** in a canonical way

symplectic manifold
(M, ω)

$\rightsquigarrow$ $\mathcal{H}$ (Hilbert space)

Brane Quantization

mirror symmetry

A-model of $\mathrm{Y} = M_{\mathbb{C}}$ $\longrightarrow$ B-model of $\widetilde{\mathrm{Y}}$

$$\mathcal{H} \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$$

$$\mathcal{H} \;=\; \mathrm{Ext}^*_{\widetilde{Y}}(\widetilde{\mathcal{B}}_{cc}, \widetilde{\mathcal{B}}')$$

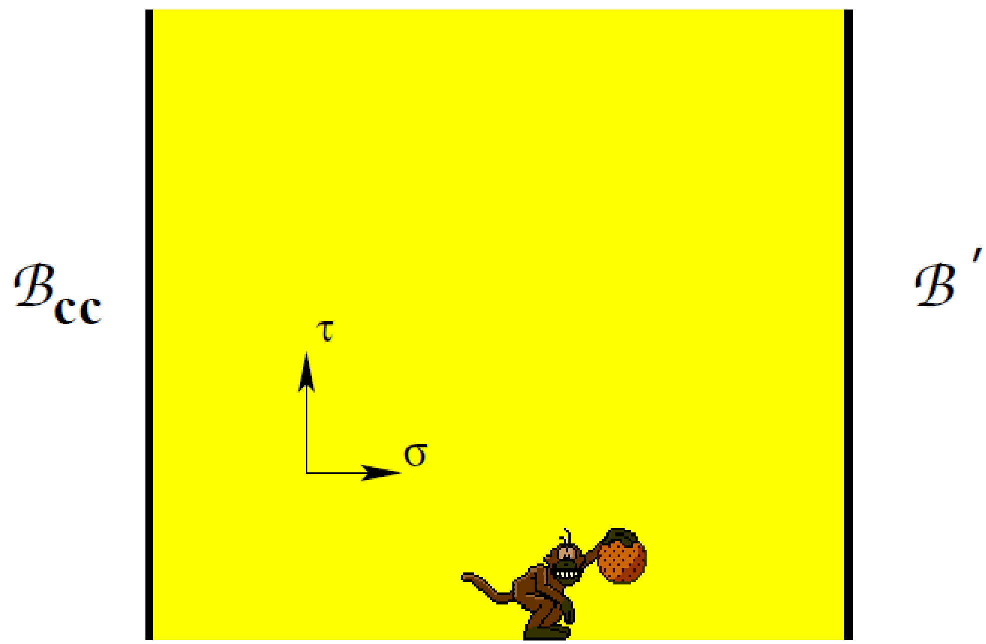
symplectic manifold
(M, ω) $\rightsquigarrow$ $\mathcal{H}$ (Hilbert space)

Brane Quantization

A-model of $Y = M_{\mathbb{C}}$:

$\mathcal{B}'$ = Lagrangian A-brane supported on

$$M \subset Y$$



$\mathcal{B}_{cc}$ = coisotropic A-brane supported on Y and endowed with a unitary line bundle $\mathcal{L}$ with a connection of curvature

$$F = \mathrm{Re}\,\Omega$$

Brane Quantization

A-model of $Y = M_{\mathbb{C}}$: $\mathcal{A}_\hbar = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$

$$\mathcal{H} = \text{space of } (\mathcal{B}_{cc}, \mathcal{B}') \text{ strings}$$

$\mathcal{B}'$ = Lagrangian A-brane supported on

$$M \subset Y$$

$\mathcal{B}_{cc}$ = coisotropic A-brane supported on Y and endowed with a unitary line bundle $\mathcal{L}$ with a connection of curvature

$$F = \mathrm{Re}\,\Omega$$

Brane Quantization

A-model of $\mathrm{Y} = M_{\mathbb{C}}$: $\mathcal{A}_\hbar \;=\; \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$

Example 1: $\mathcal{H} \;=\; \text{space of } (\mathcal{B}_{cc}, \mathcal{B}') \text{ strings}$

$M = \mathcal{O}_{\mathbb{R}}$ real coadjoint orbit of $\mathbf{G}_{\mathbb{R}}$ ⟷ $\mathcal{H} \;=$ representation of $\mathbf{G}_{\mathbb{R}}$

$\mathrm{Y} = \mathcal{O}_{\mathbb{C}}$ complex coadjoint orbit of $\mathbf{G}_{\mathbb{C}}$ $\quad \mathcal{A}_\hbar \;=\; \mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathcal{I}$

Brane Quantization

A-model of $\mathrm{Y} = M_{\mathbb{C}}$:

<u>Example 1:</u>

<u>Example 2:</u>

$M = \mathcal{M}_{\text{flat}}(G, C)$ int

... it of $\mathbf{G}_{\mathbb{R}}$

$\mathcal{H}$ = space of conformal blocks in WZW model

$\mathcal{H}$ = representation of $\mathbf{G}_{\mathbb{R}}$

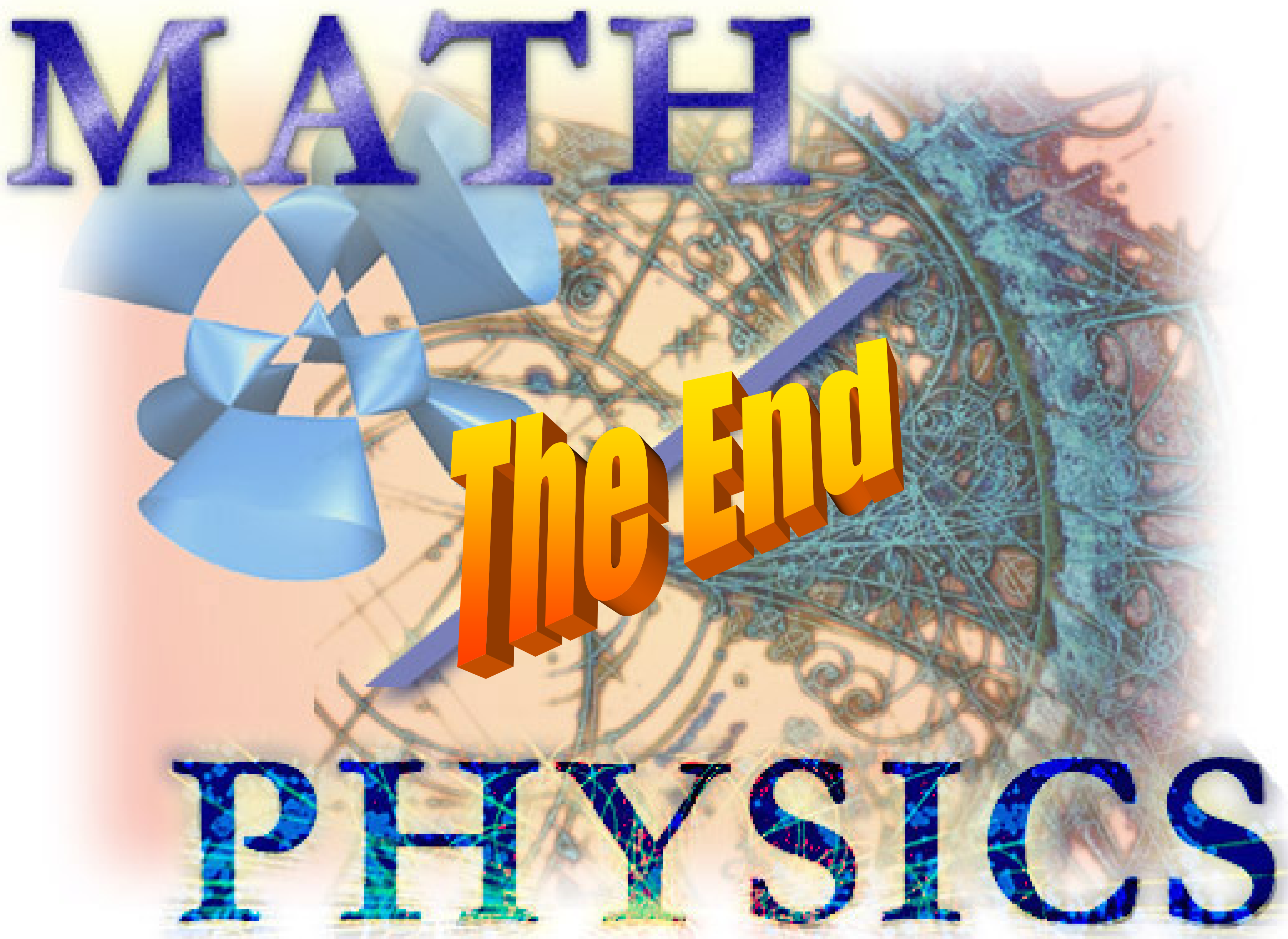
$\mathrm{Y} = \mathcal{O}$ ... plex coadjoint

o... it of $\mathbf{G}_{\mathbb{C}}$

$\mathcal{A}_\hbar$ = current algebra for the group G

$= \mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathcal{I}$

$Y = \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, C)$


MATH
The End
PHYSICS