

Double Field Theory and Duality

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Strings on a Torus



- States: momentum p , winding w
- String: Infinite set of fields $\psi(p, w)$
- Fourier transform to doubled space: $\psi(x, \tilde{x})$
- “Double Field Theory” from closed string field theory. Some non-locality in doubled space
- Subsector? e.g. $g_{ij}(x, \tilde{x})$, $b_{ij}(x, \tilde{x})$, $\phi(x, \tilde{x})$

Double Field Theory

- Double field theory on doubled torus
- General solution of string theory: involves doubled fields $\psi(x, \tilde{x})$
- DFT needed for non-geometric backgrounds
- *Real* dependence on *full* doubled geometry, dual dimensions not auxiliary or gauge artifact. Double geom. *physical* and *dynamical*

- Novel symmetry, reduces to diffeos + B-field trans. in *any* half-dimensional subtorus
- Backgrounds depending on $\{x^a\}$ seen by particles, on $\{\tilde{x}_a\}$ seen by winding modes.
- Captures exotic and complicated structure of interacting string: **Non-polynomial, algebraic structure homotopy Lie algebra, cocycles.**
- T-duality symmetry manifest
- Formalism for T-folds etc
- Generalised T-duality: no isometries needed

**Earlier work on double fields:
Siegel, Tseytlin**

Strings on Torus

$$D=n+d$$

Target space

$$\mathbb{R}^{n-1,1} \times T^d$$

Coordinates

$$x^i = (x^\mu, x^a)$$

Momenta

$$p_i = (p_\mu, p_a)$$

Winding

$$w^i = (w^\mu, w^a)$$

Dual coordinates (conjugate to winding)

$$\tilde{x}_i = (\tilde{x}_\mu, \tilde{x}_a)$$

Constant metric and B-field

$$E_{ij} = G_{ij} + B_{ij}$$

Compact dimensions

p_a, w^a discrete, in Narain lattice, x^a, \tilde{x}_a periodic

Non-compact dimensions x^μ, p_μ continuous

Usually take $w^\mu = 0$ so $\frac{\partial}{\partial \tilde{x}_\mu} = 0$, fields $\psi(x^\mu, x^a, \tilde{x}_a)$

T-Duality

- Interchanges momentum and winding
- Equivalence of string theories on dual backgrounds with very different geometries
- String field theory symmetry, provided fields depend on both x, \tilde{x} **Kugo, Zwiebach**
- For fields $\psi(x^\mu)$ not $\psi(x^\mu, x^a, \tilde{x}_a)$ **Buscher**
- Generalise to fields $\psi(x^\mu, x^a, \tilde{x}_a)$

Generalised T-duality

Dabholkar & CMH

String Field Theory on Minkowski Space

Closed SFT:
Zwiebach

String field $\Phi[X(\sigma), c(\sigma)]$

$X^i(\sigma) \rightarrow x^i$, oscillators

Expand to get infinite set of fields

$g_{ij}(x), b_{ij}(x), \phi(x), \dots, C_{ijk\dots l}(x), \dots$

Integrating out massive fields gives field theory for

$g_{ij}(x), b_{ij}(x), \phi(x)$

String Field Theory on a torus

String field $\Phi[X(\sigma), c(\sigma)]$

$X^i(\sigma) \rightarrow x^i, \tilde{x}_i$, oscillators

Expand to get infinite set of double fields

$g_{ij}(x, \tilde{x}), b_{ij}(x, \tilde{x}), \phi(x, \tilde{x}), \dots, C_{ijk\dots l}(x, \tilde{x}), \dots$

Seek **double field theory** for

$g_{ij}(x, \tilde{x}), b_{ij}(x, \tilde{x}), \phi(x, \tilde{x})$

Free Field Equations (B=0)

$$L_0 + \bar{L}_0 = 2$$

$$p^2 + w^2 = N + \bar{N} - 2$$

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

Free Field Equations (B=0)

$$L_0 + \bar{L}_0 = 2$$

$$p^2 + w^2 = N + \bar{N} - 2$$

Treat as field equation, kinetic operator in doubled space

$$G^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

$$L_0 - \bar{L}_0 = 0$$

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Treat as constraint on double fields

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} \quad (\Delta - \mu)\psi = 0$$

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$$G^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

Laplacian for metric

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

$$ds^2 = G_{ij} dx^i dx^j + G^{ij} d\tilde{x}_i d\tilde{x}_j$$

Treat as constraint on double fields

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} \quad (\Delta - \mu)\psi = 0$$

Laplacian for metric

$$ds^2 = dx^i d\tilde{x}_i$$

$$g_{ij}(x, \tilde{x}), b_{ij}(x, \tilde{x}), \phi(x, \tilde{x})$$

$$N = \bar{N} = 1$$

$$p^2 + w^2 = 0$$

$$p \cdot w = 0$$

“Double Massless”

Constrained fields $\psi(x^\mu, x^a, \tilde{x}_a)$

$$(\Delta - \mu)\psi = 0$$

Momentum space $\psi(p_\mu, p_a, w^a)$ $\Delta = p_a w^a$

Momentum space: Dimension $n+2d$

Cone: $p_a w^a = 0$ or hyperboloid: $p_a w^a = \mu$

dimension $n+2d-1$

DFT: fields on cone or hyperboloid, with discrete p, w

Problem: naive product of fields on cone do not lie on cone. Vertices need projectors

Restricted fields: Fields that depend on d of $2d$ torus

momenta, e.g. $\psi(p_\mu, p_a)$ or $\psi(p_\mu, w^a)$

Simple subsector, no projectors needed, no cocycles.

Torus Backgrounds

$$G_{ij} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix} \quad E_{ij} \equiv G_{ij} + B_{ij}$$

Fluctuations $e_{ij} = h_{ij} + b_{ij}$

Take $B_{ij} = 0$ $\tilde{\partial}_i \equiv G_{ik} \frac{\partial}{\partial \tilde{x}_k}$

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Usual action $\int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$

Quadratic part $\int dx L[h, b, d; \partial]$

$$e^{-2d} = e^{-2\phi} \sqrt{-g}$$

(d invariant under usual T-duality)

Double Field Theory Action

$$S^{(2)} = \int [dx d\tilde{x}] \left[L[h, b, d; \partial] + L[-h, -b, d; \tilde{\partial}] \right. \\ \left. + (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4d \partial^i \tilde{\partial}^j b_{ij} \right]$$

Action + dual action + strange mixing terms

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Action + dual action + strange mixing terms

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i + \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i,$$

$$\delta b_{ij} = -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i),$$

$$\delta d = -\partial \cdot \epsilon + \tilde{\partial} \cdot \tilde{\epsilon}. \quad \text{Invariance needs constraint}$$

Diffeos and B-field transformations mixed.

Invariant cubic action found for full DFT of (h,b,d)

General double fields

$$\psi(x, \tilde{x})$$

Fields on Spacetime M

$$\psi(x)$$

Restricted Fields on N, T-dual to M

$$\psi(x')$$

M, N null wrt $O(D, D)$ metric $ds^2 = dx^i d\tilde{x}_i$

Subsector with fields and parameters all restricted to M or N

- Constraint satisfied on all fields and products of fields
- No projectors or cocycles
- T-duality covariant: independent of choice of N
- Can find full non-linear form of gauge transformations
- Full gauge algebra, full non-linear action

Restricted DFT

Double fields restricted to null D-dimensional subspace N
T-duality “rotates” N to N’

Background independent fields: g,b,d:

$$\mathcal{E} \equiv E + \left(1 - \frac{1}{2} e\right)^{-1} e \quad \mathcal{E}_{ij} = g_{ij} + b_{ij}$$

O(D,D) Covariant Notation

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \quad \partial_M \equiv \begin{pmatrix} \partial^i \\ \partial_i \end{pmatrix}$$
$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad M = 1, \dots, 2D$$

Generalised T-duality transformations:

$$X'^M \equiv \begin{pmatrix} \tilde{x}'_i \\ x'^i \end{pmatrix} = h X^M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

h in $O(d,d;\mathbf{Z})$ acts on toroidal coordinates only

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}$$

$$d'(X') = d(X)$$

Buscher if fields independent of toroidal coordinates
Generalisation to case without isometries

Generalised Metric Formulation

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\mathcal{H}_{MN}, \eta_{MN}$

Generalised Metric Formulation

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2 Metrics on double space $\mathcal{H}_{MN}, \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\mathcal{H}^{MP}\mathcal{H}_{PN} = \delta^M_N$

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Covariant Transformation

$$h^P_M h^Q_N \mathcal{H}'_{PQ}(X') = \mathcal{H}_{MN}(X)$$

$$X' = hX \quad h \in O(D, D)$$

O(D,D) covariant action

$$S = \int dx d\tilde{x} e^{-2d} L$$

$$L = \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} \\ - 2 \partial_M d \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M d \partial_N d$$

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L cubic! Indices raised and lowered with η

O(D,D) covariant action

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Gauge Transformation

$$\delta_\xi \mathcal{H}^{MN} = \xi^P \partial_P \mathcal{H}^{MN} \\ + (\partial^M \xi_P - \partial_P \xi^M) \mathcal{H}^{PN} + (\partial^N \xi_P - \partial_P \xi^N) \mathcal{H}^{MP}$$

O(D,D) covariant action

$$S = \int dx d\tilde{x} e^{-2d} L$$

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Gauge Transformation

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Rewrite as “Generalised Lie Derivative”

$$\delta_\xi \mathcal{H}^{MN} = \hat{\mathcal{L}}_\xi \mathcal{H}^{MN}$$

2-derivative action

$$S = S^{(0)}(\partial, \partial) + S^{(1)}(\partial, \tilde{\partial}) + S^{(2)}(\tilde{\partial}, \tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

Gives usual action (+ surface term)

$$\int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

2-derivative action

$$S = S^{(0)}(\partial, \partial) + S^{(1)}(\partial, \tilde{\partial}) + S^{(2)}(\tilde{\partial}, \tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

Gives usual action (+ surface term)

$$\int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

$$S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde{\partial}) \quad \text{T-dual!}$$

$$S^{(1)} \quad \text{strange mixed terms}$$

- Restricted DFT:
fields independent of half the coordinates
- If independent of \tilde{x} , equivalent to usual action
- Duality covariant: duality changes which half of coordinates theory is independent of
- Equivalent to Siegel's formulation **Hohm & Kwak**
- Good for non-geometric backgrounds

Generalised Lie Derivative

$$A_{N_1 \dots}^{M_1 \dots}$$

$$\begin{aligned} \widehat{\mathcal{L}}_{\xi} A_M^N &\equiv \xi^P \partial_P A_M^N \\ &+ (\partial_M \xi^P - \partial^P \xi_M) A_P^N + (\partial^N \xi_P - \partial_P \xi^N) A_M^P \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{L}}_{\xi} A_M^N &= \mathcal{L}_{\xi} A_M^N - \eta^{PQ} \eta_{MR} \partial_Q \xi^R A_P^N \\ &+ \eta_{PQ} \eta^{NR} \partial_R \xi^Q A_M^P \end{aligned}$$

Gauge Algebra

Parameters $(\epsilon^i, \tilde{\epsilon}_i) \rightarrow \Sigma^M$

Gauge Algebra $[\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1, \Sigma_2]_C}$

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}] = -\hat{\mathcal{L}}_{[\xi_1, \xi_2]_C}$$

C-Bracket:

$$[\Sigma_1, \Sigma_2]_C \equiv [\Sigma_1, \Sigma_2] - \frac{1}{2} \eta^{MN} \eta_{PQ} \Sigma_{[1}^P \partial_N \Sigma_{2]}^Q$$

Lie bracket + metric term

Parameters $\Sigma^M(X)$ restricted to N

Decompose into vector + 1-form on N

C-bracket reduces to **Courant bracket** on N

Same covariant form of gauge algebra found in similar context by **Siegel**

Jacobi Identities not satisfied!

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [[\Sigma_1, \Sigma_2], \Sigma_3] + \text{cyclic} \neq 0$$

for both C-bracket and Courant-bracket

How can bracket be realised as a symmetry algebra?

$$[[\delta_{\Sigma_1}, \delta_{\Sigma_2}], \delta_{\Sigma_3}] + \text{cyclic} = \delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$$

Symmetry is Reducible

Parameters of the form $\Sigma^M = \eta^{MN} \partial_N \chi$
do not act

Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$

Parameters of the form $\alpha = d\beta$
do not act

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Resolution:

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^M = \eta^{MN} \partial_N \chi$$

$\delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$ does not act on fields

D-Bracket

$$[A, B]_{\text{D}} \equiv \hat{\mathcal{L}}_A B$$

$$[A, B]_{\text{D}}^M = [A, B]_{\text{C}}^M + \frac{1}{2} \partial^M (B^N A_N)$$

Not skew, but satisfies Jacobi-like identity

$$[A, [B, C]_{\text{D}}]_{\text{D}} = [[A, B]_{\text{D}}, C]_{\text{D}} + [B, [A, C]_{\text{D}}]_{\text{D}}$$

On restricting to null subspace N

C-bracket \rightarrow Courant bracket

D-bracket \rightarrow Dorfman bracket

Gen Lie Derivative \rightarrow GLD of Grana, Minasian, Petrini
and Waldram

Generalised Geometry, M-Theory

- Generalised Geometry doubles Tangent space, Metric + B-field, action of $O(D,D)$ Hitchin; Gualtieri
- DFT doubles coordinates.
- Extended geometry: extends tangent space, metric and 3-form gauge field, action of exceptional U-duality group Hull; Pacheco & Waldram
- Rewrite of 11-d sugra action in terms of extended geometry Hillman; Berman & Perry

Double Field Theory

- Constructed (unrestricted) DFT cubic action for fields g, b, d .
- Quartic & higher should have many features of string field theory: cocycles, projectors, symmetry based on a homotopy Lie algebra.
- Is there an (unrestricted) gauge-invariant action for g, b, d alone, or are massive fields needed?
- T-duality symmetry
- Stringy issues in simpler setting than SFT

- Restricted DFT: have non-linear background independent theory, duality covariant
- SUSY: Jeon, Lee, Park; Hohm & Kwak; Coimbra, Strickland-Constable, Waldram
- Type II: Hohm, Kwak, Zwiebach; Coimbra, Strickland-Constable, Waldram
- Geometry? Meaning of curvature?
- Use for non-geometric backgrounds
- General spaces, not tori?
- Doubled geometry *physical* and *dynamical*