Wedge Dislocations in the Geometric Theory of Defects

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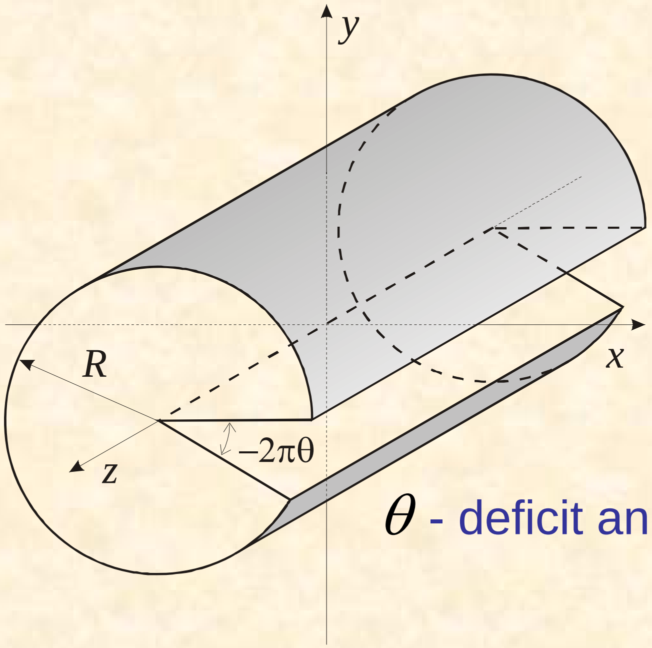
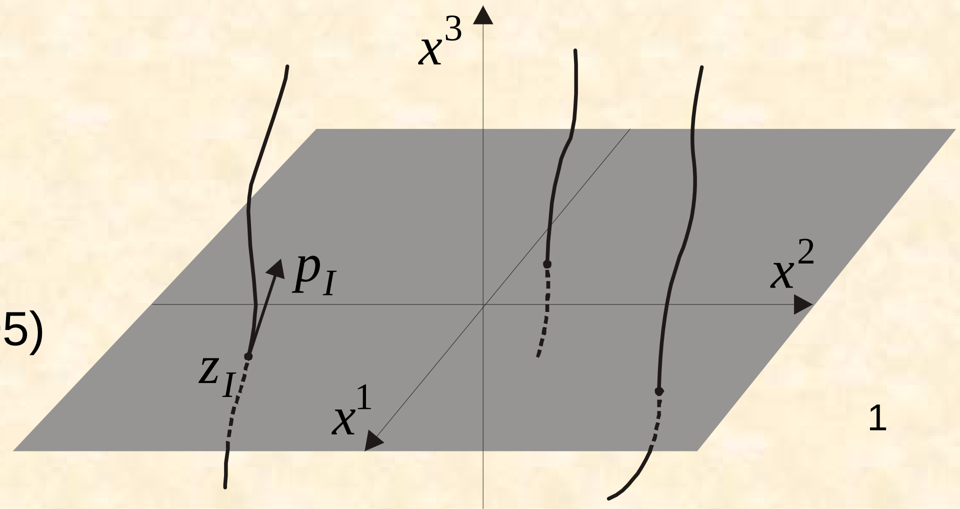
Wedge dislocation



θ - deficit angle

Staruszkiewicz (1963)
Clement (1976)
Deser, Jackiw, ‘t Hooft (1984)



Bellini, Ciafaloni, Valtancoly (1995)
Welling (1995)
Menotti, Seminara (1999)

Free energy (the action) for static distribution of wedge dislocation

$$S=\int d^3x\sqrt{g}R-\sum_I^N m_I\int d\tau\sqrt{\dot{q}_I^\alpha\dot{q}_I^\beta g_{\alpha\beta}}$$

Notations

$\mathbb{R}^3$ - continuous elastic media = Euclidean three-dimensional space

$x^\alpha \quad \alpha=1,2,3$ - global curvilinear coordinates

$g_{\alpha\beta}(x)$ - Riemannian metric

$R(g)$ - the scalar curvature

$m_I:=4\pi\theta_I$ - deficit angles

$q_I^\alpha(\tau)$ - wedge dislocation axis

$\dot{q}_I:=\dfrac{dq_I}{d\tau}$ - velosity (tangent vector)

$I=1,\dots,N$ - the number of dislocations

Equations of equilibrium

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-\frac{1}{2}T_{\alpha\beta}$$

$$\ddot{q}_I^\alpha=-\Gamma_{\beta\gamma}{}^\alpha\dot{q}_I^\beta\dot{q}_I^\gamma$$

where $T_{\alpha\beta}=\dfrac{1}{\sqrt{g}}\sum_I\dfrac{m_I\dot{q}_{I\alpha}\dot{q}_{I\beta}}{\dot{q}_I^3}\delta(\mathbf{x}-\mathbf{q})$

$$\delta(\mathbf{x}-\mathbf{q}):=\delta(x^1-q^1)\delta(x^2-q^2)$$

Canonical Formulation

$(x^1, x^2, x^3) \mapsto (x^3, x^1, x^2)$ - reordering of coordinates

$\alpha, \beta, \ldots = 3, 1, 2$

$\mu, \nu, \ldots = 1, 2$ - notations

$$g_{\alpha\beta} = \begin{pmatrix} N^2 + N^\rho N_\rho & N_\nu \\ N_\mu & g_{\mu\nu} \end{pmatrix}$$ - ADM parameterization of 3D metric

where N - lapse and N_μ - shift functions

$g_{\mu\nu}$ - 2D metric on slices $x^3 = \text{const}$

$(g_{\mu\nu}, p^{\mu\nu})$ $(q_I^\alpha, p_{I\alpha})$ - coordinates and conjugate momenta

$$S_{\text{HE}} = \int d^3x \left(p^{\mu\nu} \dot{g}_{\mu\nu} - N H_\perp^{(0)} - N^\mu H_\mu^{(0)} \right)$$ - the Hilbert-Einstein action

$$H_\perp^{(0)} = \frac{1}{\hat{e}} \left(p^{\mu\nu} p_{\mu\nu} - p^2 \right) - \hat{e}\hat{R}$$

$\hat{e} := \det g_{\mu\nu}$

$$H_\mu^{(0)} = -2\hat{\nabla}_\nu p^\nu{}_\mu$$ - general relativity constraints

$$S_I = \int d\tau \left(p_{I\alpha}\dot{q}_I^{\alpha} - \dot{q}_I^3 G_I \right)$$ - the action for wedge dislocations

$$G_I := p_{I3} - N^{\mu} p_{I\mu} + N\sqrt{m_I^2 - \hat{p}_I^2} = 0$$ - first class constraints $I = 1, \ldots, N$

$$\hat{p}_I^2 := p_{I\mu} p_{I\nu} \hat{g}^{\mu\nu}$$ $\tau_I \mapsto \tau_I'(\tau_I)$ - local invariance

The gauge $\dot{q}_I^3 = 1 \Longrightarrow$ $$S_I = \int d\tau \left(p_{I\mu}\dot{q}_I^{\mu} - N\sqrt{m_I^2 - \hat{p}_I^2} + N^{\mu} p_{I\mu} \right)$$

$p_N = 0, \quad p_{N_\mu} = 0$ - primary constraints

$$H_\perp = \frac{1}{\hat{e}}\left(p^{\mu\nu} p_{\mu\nu} - p^2 \right) - \hat{e}\hat{R} + \sum_I \sqrt{m_I^2 - \hat{p}_I^2}\, \delta(\boldsymbol{x} - \boldsymbol{q}) = 0$$

- secondary constraints

$$H_\mu = -2\hat{\nabla}_\nu p^\nu{}_\mu - \sum_I p_{I\mu}\, \delta(\boldsymbol{x} - \boldsymbol{q}_I) = 0$$

$$S_{\mathrm{T}} = \int d^3x \left(p^{\mu\nu}\dot{g}_{\mu\nu} + \sum_I p_{I\mu}\dot{q}_I^{\mu}\delta(\boldsymbol{x} - \boldsymbol{q}) - NH_\perp - N^{\mu}H_\mu \right)$$ - total Hamiltonian

Secondary constraints

$$H_\perp = \frac{1}{\hat{e}}\left(p^{\mu\nu}p_{\mu\nu} - p^2\right) - \hat{e}\hat{R} + \sum_I \sqrt{m_I^2 - \hat{p}_I^2}\,\delta(\boldsymbol{x} - \boldsymbol{q}) = 0$$

$$H_\mu = -2\hat{\nabla}_\nu p^\nu{}_\mu - \sum_I p_{I\mu}\,\delta(\boldsymbol{x} - \boldsymbol{q}_I) = 0$$

Equations for metric

$$\dot{g}_{\mu\nu} = \frac{2N}{\hat{e}} p_{\mu\nu} - \frac{2N}{\hat{e}} g_{\mu\nu} p + \hat{\nabla}_\mu N_\nu + \hat{\nabla}_\nu N_\mu,$$

$$\dot{p}^{\mu\nu} = \frac{N}{2\hat{e}}\hat{g}^{\mu\nu}\left(p^{\rho\sigma}p_{\rho\sigma} - p^2\right) - \frac{2N}{\hat{e}}\left(p^{\mu\rho}p^\nu{}_\rho - p^{\mu\nu}p\right) + \hat{e}\left(\hat{\Delta}N\hat{g}^{\mu\nu} - \hat{\nabla}^\mu\hat{\nabla}^\nu N\right)$$

$$- p^{\mu\rho}\hat{\nabla}_\rho N^\nu - p^{\nu\rho}\hat{\nabla}_\rho N^\mu + \hat{\nabla}_\rho\left(N^\rho p^{\mu\nu}\right) - N\sum_I \frac{p_I^\mu p_I^\nu}{2\sqrt{m_I^2 - \hat{p}_I^2}}\,\delta(\boldsymbol{x} - \boldsymbol{q}_I)$$

Equations for dislocations axes

$$\dot{q}_I^\mu = -\left.\frac{N}{\sqrt{m_I^2 - \hat{p}_I^2}}\right|_{\boldsymbol{x}-\boldsymbol{q}_I} p_I^\mu - N^\mu\Big|_{\boldsymbol{x}-\boldsymbol{q}_I}$$

$$\dot{p}_{I\mu} = -\partial_\mu\left[N\sqrt{m_I^2 - \hat{p}_I^2} - N^\nu p_{I\nu}\right]_{\boldsymbol{x}=\boldsymbol{q}_I}$$

Complex coordinates

$$(x^1, x^2) \mapsto (z, \overline{z}) \quad \text{where} \quad z := x^1 + ix^2, \qquad \overline{z} := x^1 - ix^2$$

Gauge fixing

$$g_{\mu\nu} = e^{2\phi}\delta_{\mu\nu}$$ - conformally flat metric $$\delta_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$p := p^{\mu\nu} g_{\mu\nu} = 0$$ - the third gauge condition

$$p^{z}{}_{\overline{z}} = p^{1}{}_{1} + ip^{1}{}_{2}, \qquad p^{\overline{z}}{}_{z} = p^{1}{}_{1} - ip^{1}{}_{2},$$

$$p^{z}{}_{z} = 0, \qquad p^{\overline{z}}{}_{\overline{z}} = 0 \qquad\qquad g_{\mu\nu},\ p^{\mu\nu} \mapsto \phi, p^{z}{}_{\overline{z}}$$

Solution of the kinematical constraints

$$H_\mu = 0 \Longrightarrow \partial_{\bar z} p^{\bar z}{}_z = -\frac{1}{2}\sum_I p_{Iz}\delta(z - z_I) \Longrightarrow p^{\bar z}{}_z = -\frac{1}{2\pi}\sum_I \frac{p_{Iz}}{z - z_I}$$

Center of mass coordinate system: $\sum_I p_{Iz} = 0$

$$p^{\bar z}{}_z = \frac{P_{N-2}(z)}{\prod_I (z - z_I)} = C\,\frac{\prod_A (z - z_A)}{\prod_I (z - z_I)}$$

where $C(z_I, p_{Iz}) := \frac{1}{2\pi}\sum_I p_{Iz}\sum_{J\neq I} z_J$

Asymptotic behavior:

$$p^{\bar z}{}_z\Big|_{z\to z_I} = -\frac{1}{2\pi}\frac{p_{Iz}}{z - z_I} \qquad p^{\bar z}{}_z\Big|_{z\to\infty} = \frac{C}{z^2}$$

Solution of the Dynamical Constraint

$$H_\perp = 0 \Longrightarrow 2\Delta\phi = 2p^{\bar z}{}_z p^z{}_{\bar z} e^{-2\phi} + \sum_I \sqrt{m_I^2 - 4p_{Iz}p_{I\bar z}e^{-2\phi}}\ \delta(z - z_I)$$

$$2\tilde\phi := 2\phi - \ln\left(2p^{\bar z}{}_z p^z{}_{\bar z}\right)$$ - ansatz

$$2\Delta\tilde\phi = e^{-2\tilde\phi} + 4\pi\sum_I (a_I + 1)\ \delta(z - z_I) - 4\pi\sum_A \delta(z - z_A)$$

where $4\pi a_I := \sqrt{m_I^2 - \hat p_I^2}$

Theorem: $\hat p_I^2(z_I) = 0$

$$2\Delta\tilde\phi = e^{-2\tilde\phi} + 4\pi\sum_I (\theta_I + 1)\ \delta(z - z_I) - 4\pi\sum_A \delta(z - z_A)$$

Asymptotic behavior:

$$e^{2\phi}\Big|_{z\to z_I} \sim \left[\left(z - z_I\right)\left(\bar z - \bar z_I\right)\right]^{-|\theta_I|}$$

$$M = \int_{x^3=\text{const}} d^2x\,\hat e\hat R = 2\int_{x^3=\text{const}} dz\,\Delta\phi = \text{const} < \infty$$ - the Euler nimber

$$e^{2\phi}\Big|_{z\to\infty} \simeq \left(z\bar z\right)^\mu \qquad \mu := \frac{M}{4\pi} - \sum|\theta_I| \qquad \mu > -1$$

Lapse function

$$\dot{p} = \dot{p}^{\mu\nu} g_{\mu\nu} + p^{\mu\nu} \dot{g}_{\mu\nu} = 0 \quad \Longrightarrow$$

$$\hat{e}\Delta N + \frac{N}{\hat{e}} p^{\mu\nu} p_{\mu\nu} - N \sum_I \frac{\hat{p}_I^2}{m_I} \delta\left(\boldsymbol{x} - \boldsymbol{q}_I \right) = 0$$

$$\Downarrow$$

$$\Delta N = -2e^{-2\tilde{\phi}} N - e^{-2\phi} N \sum_I \frac{4 p_{Iz} p_{I\bar{z}}}{\left| m_I \right|} \delta\left(z - z_I \right)$$

One can prove that $N(z_I)$ is finite

$$\Delta N = -e^{-2\tilde{\phi}} N \quad \Longrightarrow \quad \boxed{N = \frac{\partial\left(2\tilde{\phi} \right)}{\partial M}}$$

- general solution

Asymptotic behavior:

$$N\Big|_{z \to z_I} \simeq c_I - 2 \left(\frac{\left(z - z_I \right)\left(\bar{z} - \bar{z}_I \right)}{\text{const}} \right)^{\left| \theta_I \right|} \qquad c_I \neq 0$$

$$N\Big|_{z \to \infty} \simeq \frac{1}{4\pi} \ln\left(z\bar{z} \right)$$

Shift functions

$$\dot{g}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (g^{\rho\sigma} \dot{g}_{\rho\sigma}) = 0$$

- traceless part of the equation $\dot{g}_{\mu\nu} = 0$

$$\Downarrow$$

$$\frac{2N}{\hat{e}} p_{\mu\nu} + \hat{\nabla}_\mu N_\nu + \hat{\nabla}_\nu N_\mu - g_{\mu\nu} \hat{\nabla}_\rho N^\rho = 0$$

$$\Downarrow$$

$$\partial_{\bar{z}} N^z = -N e^{-2\phi} p^z{}_{\bar{z}} \qquad \times p^z{}_{\bar{z}}$$

$$\Downarrow$$

$$p^{\bar{z}}{}_z \partial_{\bar{z}} N^z = 2 \partial_{\bar{z}} \partial_z N$$

$$\Downarrow$$

$$N^z = \frac{2}{p^{\bar{z}}{}_z} \partial_z N + g(z)$$

- general solution

$$p^{\bar{z}}{}_z = C \frac{\prod_A (z - z_A)}{\prod_I (z - z_I)} \qquad g(z) \sim \frac{P(z)}{\prod_A (z - z_A)}$$

- meromorphic function

Solution of Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{2\sqrt{g}} \sum_I \frac{m_I \dot{q}_{I\alpha} \dot{q}_{I\beta}}{\dot{q}_I^3} \delta(\boldsymbol{x} - \boldsymbol{q}_I)$$

The gauge $g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}$ $\quad p := p^{\mu}{}_{\mu} = 0$

Center of mass coordinate system: $\sum_I p_{Iz} = 0$

Momenta $p^{\bar{z}}{}_{z} = C \frac{\prod_A (z - z_A)}{\prod_I (z - z_I)}$ where $C(z_I, p_{Iz}) := \frac{1}{2\pi} \sum_I p_{Iz} \sum_{J \neq I} z_J$

The conformal factor $e^{2\phi} = 2 p^{\bar{z}}{}_{z} p^{z}{}_{\bar{z}} e^{2\tilde{\phi}}$

$$2\Delta\tilde{\phi} = e^{-2\tilde{\phi}} + 4\pi \sum_I (\theta_I + 1)\, \delta(z - z_I) - 4\pi \sum_A \delta(z - z_A)$$ - the Liouville equation

The lapce function $N = \frac{\partial\left(2\tilde{\phi}\right)}{\partial M}$

Shift functions $N^z = \frac{2}{p^{\bar{z}}{}_{z}} \partial_z N + g(z)$

Dislocations axes

Equilibrium equations:

$$\dot{q}_I^{\mu} = -\frac{N}{\sqrt{m_I^2 - \hat{p}_I^2}} p_I^{\mu} - N^{\mu}$$

$$\dot{p}_{I\mu} = -\partial_{\mu}\left[N\sqrt{m_I^2 - \hat{p}_I^2} - N^{\nu} p_{I\nu} \right]_{\boldsymbol{x}=\boldsymbol{q}_I}$$

$$\Downarrow$$

$$\dot{z}_I = -g(z_I)$$

$$\dot{p}_{Iz} = -p_{I\mu} \left.\frac{\partial N_{\mu}}{\partial z}\right|_{z=z_I} + m_I \left.\frac{\partial N}{\partial z}\right|_{z=z_I}$$

The Liouville equation

$$2\Delta\tilde{\phi} = e^{-2\tilde{\phi}} + 4\pi \sum_I (\theta_I + 1)\ \delta(z - z_I) - 4\pi \sum_A \delta(z - z_A)$$

For simlicity consider the case of two dislocations $I = 1, 2$

The solution $e^{-2\tilde{\phi}} = \dfrac{8 f'(z) \bar{f}'(\bar{z})}{\left(1 + f(z)\bar{f}(\bar{z})\right)^2}$ where

$$f(z) = \frac{y^1(z)}{y^2(z)} \qquad y^a(z), \qquad a = 1, 2$$

- two independent solutions of Fuchsian second order differential equation

$$y^a(z)\Big|_{z \to z_I} \sim (z - z_I)^{\beta_{aI}} \qquad \beta_{aI}$$

- local exponents

$$f(z)\Big|_{z \to z_I} \sim (z - z_I)^{\beta_{1I} - \beta_{2I}}$$

$$\beta_{2I} - \beta_{1I} = \theta_I$$

The Fucksian differential equation

$$y'' + Q(z)y = 0$$

$$Q(z) = \frac{1}{4}\left[\frac{1-\theta_1^2}{(z-z_1)^2} + \frac{1-\theta_2^2}{(z-z_2)^2} + \frac{\theta_1^2+\theta_2^2-\theta_\infty^2-1}{(z-z_1)(z-z_2)}\right]$$

The Riemann scheme
$$\begin{pmatrix} z_1 & z_2 & z_\infty \\ \frac{1-\theta_1}{2} & \frac{1-\theta_2}{2} & \frac{-1-\theta_\infty}{2} \\ \frac{1+\theta_1}{2} & \frac{1+\theta_2}{2} & \frac{-1+\theta_\infty}{2} \end{pmatrix}$$

$$z \mapsto \varsigma = \frac{z-z_1}{z_2-z_1} \qquad z_1, z_2, z_\infty \mapsto 0, 1, \infty$$

$$\varsigma(1-\varsigma)y'' + \left[c-\left(a+b+1\right)\varsigma\right]y' - ab\, y = 0$$ - hypergeometric equation

The Riemann scheme
$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \beta_{11}+\beta_{12}+\beta_{1\infty} \\ \beta_{21}-\beta_{11} & \beta_{22}-\beta_{12} & \beta_{11}+\beta_{12}+\beta_{2\infty} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{pmatrix}$$

Conclusion

1 Arbitrary distribution of wedge dislocations in elastic media is discribed within the Geometric theory of defects.

2 The free energy is given by 3-dimensional Euclidean gravity coupled to point particles.

2 Einstein’s equations are reduced to solving the Fuchsian differential equation

3 For two wedge dislocations the problem is solved analytically in terms of the hypergeometric functions

Reduction to the Riemann—Hilbert problem

$$g_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \delta_{ab}$$

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\varepsilon}\varepsilon_{\gamma\delta\varsigma}R^{\varepsilon\varsigma}$$

$\varepsilon_{\alpha\beta\gamma}$ - totally antisymmetric tensor

$$R_{\alpha\beta\gamma\delta} = 0 \quad \Longrightarrow \quad e_\alpha{}^a = \partial_\alpha y^a$$

The gauge: $S = \int d^2 z \sqrt{\det h}\, h^{\alpha\beta}\partial_\alpha y^a \partial_\beta y_a \quad \Longrightarrow \quad \partial_z \partial_{\overline{z}} y^a = 0$

$$\Downarrow$$

$$y^a = F^a(z, x^3) + G^a(\overline{z}, x^3) + H^a(x^3)$$

$e_z{}^a := \partial_z y^a = e_z{}^a(z, x^3)$ - holomorphic

$e_{\overline{z}}{}^a := \partial_{\overline{z}} y^a = e_{\overline{z}}{}^a(\overline{z}, x^3)$ - antiholomorphic $\qquad e_{\overline{z}}{}^a = \overline{e_z{}^a}$

$$e_3{}^a := \partial_3 y^a = C^a(x^3) + \int dz\, e_z{}^a + \int d\overline{z} e_{\overline{z}}{}^a$$

Reduction to the Riemann—Hilbert problem

Everything is defined by $e_z{}^a(z, x^3)$

Let γ_I be a closed loop around the dislocation axis at z_I

$$y^a(z_0) \mapsto \tilde{y}(z_0) = \int_{z_0}^{z_I} dz\, e_z{}^a + \oint_{\gamma_I} dz\, e_z{}^a + \int_{z_I}^{z_0} dz\, e_z{}^a$$

$$= (1 - M_I) Y_I^a + \oint_{\gamma_I} dz\, e_z{}^a$$

$M_I \in \mathbb{SO}(3)$ - the monodromy matrix

$$\pi\left(\mathbb{C} \setminus \{z_1, \ldots, z_N, z_\infty\}; z_0\right) \to \mathbb{SO}(3) \subset \mathbb{GL}(3, \mathbb{C})$$

the Riemann—Hilbert problem

Show that for any representation $\pi\left(\mathbb{C} \setminus \{z_1, \ldots, z_N, z_\infty\}; z_0\right) \to \mathbb{GL}(p, \mathbb{C})$

there is a Fuchsian system of equations with a given monodromy