Reparametrization path integral in AdS and the holographic Schwinger effect

Yuri Makeenko (ITEP, Moscow)

Based on:

- J. Ambjørn, Y. M. Phys. Rev. D 85, 061901 (2012) [arXiv:1112.5606]
- C. Kristjansen, Y.M. to appear

Extending to SYM the previous applications of the reparametrization path integral to QCD/string

Y. M., Poul Olesen

- Phys. Rev. Lett. **102**, 071602 (2009) [arXiv:0810.4778 [hep-th]]
- Phys. Rev. D **80**, 026002 (2009) [arXiv:0903.4114 [hep-th]]
- Phys. Rev. D **82**, 045025 (2010) [arXiv:1002.0055 [hep-th]]
- JHEP **08**, 095 (2010) [arXiv:1006.0078 [hep-th]]
- Phys. Lett. B **709**, 285 (2012) [arXiv:1111.5606 [hep-th]]

P. Buividovich, Y.M.

- Nucl. Phys. B **834**, 453 (2010) [arXiv:0911.1083 [hep-th]]

Y. M.

- Phys. Rev. D **83**, 026007 (2011) [arXiv:1012.0708 [hep-th]]
- Phys. Lett. B **699**, 199 (2011) [arXiv:1103.2269 [hep-th]]

Contents of the talk

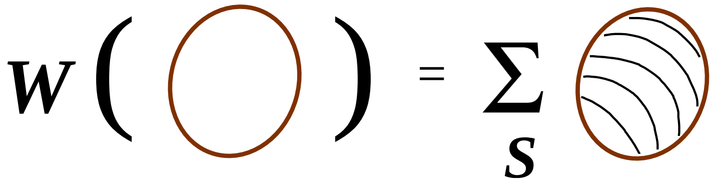
- circular Wilson loop in AdS/CFT
  - the Poisson formula in the Lobachevsky plane
  - Douglas' integral for circular loop in AdS
- accounting for semiclassical fluctuations about the minimal surface
  - reparametrization path integral
  - IIB superstring in $AdS_5 \times S^5$
- application to the Schwinger effect in $\mathcal{N} = 4$ SYM
  - shifting the critical electric field at strong coupling

AdS/CFT for Wilson Loops

Maldacena (1998)
Rey, Yee (1998)

Wilson loop in $\mathcal{N}=4$ SYM = IIB open superstring in $AdS_5 \times S^5$

$$W_{\text{SYM}}(C) = \sum_{S:\partial S = C} \mathrm{e}^{-A_{IIB \text{ on } AdS_5 \otimes S^5}}$$



$$C = (x^\mu(\sigma), \int^\sigma \mathrm{d}\sigma\, |\dot{x}| n^i)$$

— loop in the boundary of $AdS_5 \otimes S^5$
e.g. $n^i = (1,0,0,0,0,0) \Rightarrow$ 4D contour $x^\mu(\sigma)$

Circular loop:

AdS (supergravity) — Berenstein, Corrado, Fischler, Maldacena (1998); Drukker, Gross, Ooguri (1999)

CFT (exact) — Erickson, Semenoff, Zarembo (2000); Drukker, Gross (2001)

(perfect agreement !!!)

Minimal surface in AdS for circular loop

Berenstein, Corrado, Fischler, Maldacena (1998)

Drukker, Gross, Ooguri (1999)

Upper half-plane (UHP) parametrization of the surface:
$z = x + \mathrm{i}y$ $(y > 0)$ is customary in string theory.
Standard embedding space coordinates Y_{-1}, Y_0, Y_1, Y_2, Y_3, Y_4 obey

$$Y \cdot Y \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1 \qquad (1)$$

The Euler–Lagrange equations in the embedding Y-space are

$$(-\Delta + 2)\, Y_i = 0, \qquad \Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and the "mass" 2 arises because of the presence of the Lagrange multiplier which is used to implement Eq. (1).

Solution for the minimal surface in AdS for circular boundary:

$$Y_1 = \frac{1 - x^2 - y^2}{2y} \quad Y_2 = \frac{x}{y} \quad Y_{-1} = \frac{1 + x^2 + y^2}{2y} \quad Y_4 = Y_0 = Y_3 = 0$$

Minimal surface in AdS for circular loop (cont.)

On the Poincare patch

$$Z \equiv \frac{R}{Y_{-1} - Y_4} = \frac{2y\,R}{1+x^2+y^2}$$

$$X_1 \equiv ZY_1 = \frac{1-x^2-y^2}{1+x^2+y^2}R \qquad X_2 \equiv ZY_2 = \frac{2x\,R}{1+x^2+y^2}$$

is a sphere $X_1^2 + X_2^2 + Z^2 = R^2$ with a circular boundary for $Z = 0$.

The induced metric

$$\mathrm{d}\ell^2 \equiv \mathrm{d}Y \cdot \mathrm{d}Y = \frac{\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \mathrm{d}Z^2}{Z^2} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

is the Poincare metric of the Lobachevsky plane.

Dirichlet Green function in AdS

Extension of Douglas' (1931) algorithm for finding minimal surfaces to the Lobachevsky plane:

- to construct the Dirichlet Green function on the Lobachevsky plane
- to derive the Poisson formula for the Lobachevsky plane.

This will reconstruct the minimal surface from its boundary value – finding the minimal surface is reduced to minimizing a boundary functional with respect to reparametrizations.

Dirichlet Green function on the Lobachevsky plane depends on the (geodesic) distance between images of the points (x_1, y_1) and (x_2, y_2):

$$L^2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4y_1 y_2}.$$

Acting by the operator $(-\Delta + 2)$, we obtain the Legendre equation whose solution for the Dirichlet Green function is

$$G = -\frac{3}{4\pi}\left(\frac{(x_1 - x_2)^2 + y_1^2 + y_2^2}{4y_1 y_2} \ln \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{(x_1 - x_2)^2 + (y_1 + y_2)^2} + 1\right) \qquad (2)$$

Poisson formula in AdS

Poisson formula reconstructs a harmonic function in the Lobachevsky plane from its boundary value. We take the normal derivative of Eq. (2) near the boundary at a certain minimal value $y_2 = y_{\text{min}}$ to regularize divergences:

$$\left.\frac{\partial G\left(x_1, y_1; x_2, y_2\right)}{\partial y_2}\right|_{y_2 = y_{\text{min}}} = \frac{2 y_1^2 y_{\text{min}}}{\pi((x_1 - x_2)^2 + y_1^2)^2} + \mathcal{O}(y_{\text{min}}^3).$$

Finally, we obtain

$$Y_i(x, y) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}s}{\pi} \frac{2 Y_i(t(s)) y^2 y_{\text{min}}}{((x - s)^2 + y^2)^2} \qquad (3)$$

where $Y_i(t(s))$ is the boundary value and $t(s)$ ($\mathrm{d}t/\mathrm{d}s \geq 0$) is a possible reparametrization of the boundary – crucial in Douglas' algorithm.

This extends the Poisson formula to the Lobachevsky plane.

Poisson formula in AdS (cont.)

The above spherical solution is reproduced by Eq. (3) from the boundary values

$$\begin{aligned} Y_1(t) &= \frac{1-t^2}{2y_{\text{min}}} \quad Y_2(t) = \frac{t}{y_{\text{min}}} \quad Y_{-1}(t) = \frac{1+t^2}{2y_{\text{min}}} \\ Y_0(t) &= Y_3(t) = Y_4(t) = 0 \end{aligned} \tag{4}$$

for $t(s) = s$, which means that no reparametrization of the boundary is required for a circle, in analogy with flat plane.
This is because the coordinates in use are conformal for a circle.
Note that y_{min} is nicely canceled, when (4) in substituted in Eq. (3).

An extension of Douglas' functional to AdS

Douglas integral in flat space

$$S_{\text{flat}} = \frac{1}{4\pi} \int \mathrm{d}s_1 \int \mathrm{d}s_2 \, \frac{(x_B(t(s_1)) - x_B(t(s_2)))^2}{(s_1 - s_2)^2} \tag{5}$$

to be minimized with respect to the functions $t(s)$, reparametrizing the boundary. The minimization is required for $X(x,y)$ to obey a conformal gauge, where the Nambu–Goto would coincide with the quadratic (Polyakov) action.

Douglas integral in AdS space $S_{\text{AdS}} = S_{\text{div}} + S_{\text{reg}}$ Ambjørn, Y.M. (2012)

$$S_{\text{reg}} = \frac{1}{2\pi} \int \mathrm{d}s_1 \int \mathrm{d}s_2 (Y_B(t(s_1)) - Y_B(t(s_2)))^2 y_{\text{min}}^2 \left[\frac{1}{(s_1 - s_2)^4}\right]_{\text{reg}} \tag{6}$$

$$\left[\frac{1}{(s_1 - s_2)^4}\right]_{\text{reg}} = \left[\frac{1}{((s_1 - s_2)^2 + 4y_{\text{min}}^2)^2} + \frac{32 y_{\text{min}}^2}{((s_1 - s_2)^2 + 4y_{\text{min}}^2)^3} - \frac{384 y_{\text{min}}^4}{((s_1 - s_2)^2 + 4y_{\text{min}}^2)^4}\right].$$

This boundary functional to be minimized with respect to $t(s)$.

Regularization by shifting the boundary

Integral in Eq. (6) is like in Eq. (5), while the denominator in Eq. (6) is $(s_1 - s_2)^4$ rather than $(s_1 - s_2)^2$ as in Eq. (5). This results in the well-known UV divergences regularized by shifting the boundary from $y = 0$ to $y = y_{\rm min}$. In the dual language of D-branes this corresponds to the breaking Maldacena (1998), Rey, Yee (1998)
$U(N) \longrightarrow U(N-1) \times U(1)$ by assigning a finite mass to the $U(1)$ gauge boson. This mass is associated with shifting the boundary to the slice $Z = \varepsilon$, so that from Eq. (4)

$$y_{\rm min}(t) = \frac{\varepsilon}{2R}(t^2 + 1)$$

The divergent part

$$S_{\rm div} = 2\pi \frac{R - \varepsilon}{\varepsilon},$$

comes from $(s_1 - s_2) \sim y_{\rm min}$ and does not depend on $t(s)$. The regularized part $S_{\rm reg}$ now gives a finite contribution in view of the important formula

$$\int {\rm d}s\, s^2 \left[\frac{1}{s^4}\right]_{\rm reg} = 0.$$

Reparametrization path integral in $\mathcal{N}=4$ SYM

Reparametrization path integral for the circular Wilson loop in $\mathcal{N}=4$ SYM

$$W\,(\text{circle}) = \mathrm{e}^{-\sqrt{\lambda}S_{\text{div}}/2\pi} \int \mathcal{D}_{\text{diff}} t(s)\, \mathrm{e}^{-\sqrt{\lambda}S_{\text{reg}}[t]/2\pi}, \tag{7}$$

where

$$S_{\text{reg}}[t] = \frac{1}{2\pi}\int \mathrm{d}s_1 \mathrm{d}s_2 \,(t(s_1)-t(s_2))^2 \left[\frac{1}{(s_1-s_2)^4}\right]_{\text{reg}}$$

since S_{div} does not depend on the reparametrization.

The constant $\sqrt{\lambda}$ is prescribed by the AdS/CFT correspondence

$$\sqrt{\lambda} = \frac{R^2_{AdS}}{\alpha'}$$

but we consider it as a parameter to be fixed by comparing with the $\mathcal{N}=4$ SYM Wilson loop.

Expanding the reparametrizing function $t(s) = s + \frac{1}{\sqrt[4]{\lambda}}\beta(s)$ we have

$$\sqrt{\lambda}S_{\text{reg}} = \frac{1}{2\pi}\int \mathrm{d}s_1 \mathrm{d}s_2 \,(\beta(s_1)-\beta(s_2))^2 \left[\frac{1}{(s_1-s_2)^4}\right]_{\text{reg}} \tag{8}$$

Reparametrization path integral in $\mathcal{N}=4$ SYM (cont.)

The action is exact, but we expand in $1/\sqrt[4]{\lambda}$ to quadratic order because the measure for integrating over subordinated functions with $\mathrm{d}t(s)/\mathrm{d}s \geq 0$ is highly nonlinear. Only to the quadratic order it can be substituted by the ordinary Lebesgue measure.

The integral (8) has three zero modes

$$\beta_1(s) = 1, \quad \beta_2(s) = s, \quad \beta_3(s) = s^2$$

which is a consequence of three $SL(2,\mathbb{R})$ symmetries.

These result in a preexponential factor of $\lambda^{-3/4}$ in a full analogy with the string theory analysis Drukker, Gross (2001)
We thus obtain from the ansatz (7) at large λ:

$$W\,(\text{circle}) \propto \lambda^{-3/4}\, \mathrm{e}^{\sqrt{\lambda}}$$

reproducing the result Erickson, Semenoff, Zarembo (2000)
for the $\mathcal{N}=4$ SYM perturbation theory, providing λ is identified with the 't Hooft coupling.

Mass-dependence of the effective action

We have consider so far the λ-dependence of the one-loop effective action rather than its dependence on the $U(1)$ boson mass

Maldacena (1998), Rey, Yee (1998)

$$m = \frac{\sqrt{\lambda}}{2\pi\varepsilon}$$

The calculation is pretty much similar to that of Olesen, Y.M. (2010) for a $T \times R$ rectangle in flat space, where the Lüscher term was obtained from the reparametrization path integral In that case T/R was large, now R/ε is large.

The computation is performed by a mode expansion

$$\beta(s) = \sum_n \beta_n f_n(s) \qquad f_{-n}(s) = f_n^*(s)$$

using a complete set of orthogonal (complex) basis functions $f_n(s)$ and then doing the Gaussian integrals over β_n's.

Mass-dependence of the effective action (cont.)

Restricting ourselves by those modes for which the integral (8) has maximal "divergence" $\sim (R/\varepsilon)^{\nu}$, we obtain Ambjørn, Y.M. (2012)

$$\prod_n \left(\frac{R}{\varepsilon}\right)^{-\nu/2} = \left(\frac{R}{\varepsilon}\right)^{\nu/2} = \mathrm{e}^{\frac{\nu}{2}\ln(R/\varepsilon)}$$

where the product goes over those modes for which the integral (8) is $\sim (R/\varepsilon)^{\nu}$ and the product is understood via the ζ-function regularization.

The value of ν is determined by the Hausdorff dimension of typical trajectories in the reparametrization path integral which is zero

Buividovich, Y.M. (2010)

This corresponds to

$$\nu = 3$$

Circular loop and the Schwinger effect

Saddle-point (Euclidean) action determining the exponent of the production rate in a constant electric field is given by the minimum of

$$S = 2\pi R m - \pi |eE| R^2 - \ln W \text{ (circle)}$$

with respect to the radius R of the circle. This effective action emerges after performing the path integral over (pseudo)particle trajectories, representing the vacuum-to-vacuum amplitude in an external constant electric field.

In the path integral, first the integral over the proper time has a saddle point, and then the saddle-point trajectory is a circle of (large) radius $R = m/|eE|$ Affleck, Alvarez, Manton (1982)

The circle lies in the μ,ν-plane, when the constant electric field E is the μ,ν-component of the field strenght $F_{\mu\nu}$.

The existence of this saddle point is justified for small $|eE|$, when the logarithm of the Wilson loop on the right-hand side is subleading at weak couplings and contributes only to the preexponential.

Holographic Schwinger effect in $\mathcal{N}=4$ SYM

Gorsky, Saraikin, Selivanov (2002)

Holographic description of the Schwinger effect in SYM:
In the gravity approximation the minimal surface does not fluctuate, so the classical action reads

$$\sqrt{\lambda} S_{\text{cl}} = \sqrt{\lambda}\pi \left(\cosh\rho - 1 - \frac{|eE|}{m^2} \sinh^2\rho \right)$$

where $\sinh\rho = R/\varepsilon = 2\pi m R/\sqrt{\lambda}$. This formula is applicable for $|eE| \lesssim m^2$, when the minimization of S_{cl} with respect to ρ gives

$$\cosh\rho_0 = \frac{2\pi m^2}{|eE|\sqrt{\lambda}} \qquad (9)$$

This equation has no solution for ρ_0 when $|eE| > 2\pi m^2/\sqrt{\lambda}$, which implies the existence of a critical electric field like in string theory. Semenoff, Zarembo (2011)

How fluctuations about the minimal surface affect this very interesting result?

Schwinger effect in $\mathcal{N} = 4$ SYM (cont.)

Ambjørn, Y.M. (2012)

For the sum of S_{cl} plus the contribution from fluctuations about the minimal surface in the quadratic approximation we have

$$\sqrt{\lambda} S_{\text{cl+1loop}} = \sqrt{\lambda}\pi \left(\cosh\rho - 1 - \frac{|eE|}{m^2} \sinh^2\rho \right) - \frac{\nu}{2} \ln\cosh\rho \tag{10}$$

The negative sign in the second line of this formula is like for the Lüscher term in string theory.

The minimum of the effective action (10) is now reached for

$$\frac{1}{\cosh\rho_0} = \frac{\sqrt{\lambda}}{\nu} \left(1 - \sqrt{1 - \frac{\nu|eE|}{\pi m^2}} \right) \tag{11}$$

so the solution (9) is only slightly modified by quantum fluctuations. They shift the critical value of the constant electric field to

$$|eE_c| = \pi m^2 \left(\frac{2}{\sqrt{\lambda}} - \frac{\nu}{\lambda} \right)$$

where $\nu = 3$. Thus the quantum fluctuations about the minimal surface result in a $1/\sqrt{\lambda}$ correction at large λ, as it might be expected.

Fluctuations of open superstring in $AdS_5 \times S^5$

Kristjansen, Y.M. (2012)

Fluctuations about the minimal surface result at one loop in the ratio of the determinants Drukker, Gross, Tseytlin (2000)

$$Z^{(1)}_{\text{AdS}} = \frac{\det\left(-\Delta^{\text{gh}}_{ij} + \delta_{ij}\right)^{1/2}_{\text{ghost}}}{\det\left(-\Delta_{ij} + \delta_{ij}\right)^{1/2}_{\text{long.}}} \frac{\det\left(-\hat{\Delta} + R^{(2)}/4 + 1\right)^{8/2}_{\text{Fermi}}}{\det\left(-\Delta + 2\right)^{3/2}_{\text{Bose}} \det\left(-\Delta\right)^{5/2}_{\text{Bose}}}$$

The ratio of ghost to longitudinal dets is generically not 1 because of different boundary conditions.

The strategy is to assume $Z^{(1)}_{\text{flat}} = 1$ and calculate the ratio

$$\frac{Z^{(1)}_{\text{AdS}}}{Z^{(1)}_{\text{flat}}} = \frac{\det\left(-\Delta\right)}{\det\left(-\Delta_{ij} + \delta_{ij}\right)^{1/2}} \left(\frac{\det\left(-\hat{\Delta} + R^{(2)}/4 + 1\right)}{\det\left(-\hat{\Delta} + R^{(2)}/4\right)}\right)^{8/2} \left(\frac{\det\left(-\Delta\right)}{\det\left(-\Delta + 2\right)}\right)^{3/2}$$

of massive to massless dets noting that the ghost dets are the same.

Fluctuations of open superstring in $AdS_5 \times S^5$ (cont.)

Every ratio of massive to massless dets is computable either by the Seeley coefficients (modulo a constant) Drukker, Gross, Tseytlin (2000) or by direct computation of 1D×angular dets Kruczenski, Tirziu (2008)

The structure that appears in the log of the ratio is like

$$\ln \frac{\det(-\Delta+\mu^2)}{\det(-\Delta)} = -\mu^2 \frac{1}{4\pi} \int \sqrt{g} \ln \sqrt{g} = \mu^2 \frac{1}{2\pi\varepsilon} (\ln \Lambda\varepsilon + 1) + \text{const.}$$

with the total coefficient (extracted from the Seeley coefficients) 2×1 (longitudinal) + 3×2 (transversal) - 8×1 (GS fermions) =0

Therefore $Z^{(1)}_{\text{AdS}} = \text{const.}$ does not depend on ε.

Like in the flat space the Liouville field $\varphi(x,y)$ ($g_{ab} = \mathrm{e}^{\varphi}\delta_{ab}$) decouples in the bulk, while its boundary value is related to the reparametrizing function $t(s)$ as

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \mathrm{e}^{\varphi(s,0)/2}$$

We are thus left with the same boundary action as previously discussed (= AdS Douglas' integral), reproducing the same effective action.

Fluctuations of open superstring in $AdS_5 \times S^5$ (cont.)

const. is calculable by the Gel'fand–Yaglom method:

$$\frac{\det\left(-\partial^2 + V_1(x)\right)}{\det\left(-\partial^2 + V_2(x)\right)} = \frac{\Psi_1(\infty)}{\Psi_2(\infty)} \qquad \Psi_i(\varepsilon) = 0, \quad \Psi_i'(\varepsilon) = 1$$

Straight line:

$$\begin{aligned}
\frac{\det^{8/2}(\text{AdS Fermi})}{\det^{8/2}(\text{free Fermi})} &= \prod_\omega \left(\frac{1+\frac{1}{2\varepsilon\omega}}{-2\varepsilon\omega\, e^{-2\varepsilon\omega}\text{Ei}(-2\varepsilon\omega)}\right)^4 = e^{\frac{4}{\varepsilon}\ln(\Lambda\varepsilon)+\ldots} \\
\frac{\det^{3/2}(\text{AdS Bose})}{\det^{3/2}(\text{free Bose})} &= \prod_\omega \left(1+\frac{1}{\varepsilon\omega}\right)^3 = e^{\frac{3}{\varepsilon}\ln(\Lambda\varepsilon)+\ldots} \\
\frac{\det^{1/2}(\text{longitudinal})}{\det(\text{free Bose})} &= \prod_\omega \left(1+\frac{1}{\varepsilon\omega}+\frac{1}{2\varepsilon^2\omega^2}\right) = e^{\frac{1}{\varepsilon}\ln(\Lambda\varepsilon)+\ldots}
\end{aligned} \tag{12}$$

$$\frac{Z^{(1)}_{\text{AdS}}}{Z^{(1)}_{\text{flat}}} = e^{(4-3-1)\frac{1}{\varepsilon}\ln(\Lambda\varepsilon)+\ldots}$$

Same results for the most singular part for the circle

– essential deviation from Kruczenski, Tirziu (2008)

Conclusion and Outlook

- Douglas' algorithm for constructing minimal surfaces can be extended to AdS
  – explicitly elaborated for circular loop

- reparametrization path integral accounts for semiclassical fluctuations about the minimal surface at one loop
  – same results as for IIB open superstring in $AdS_5 \times S^5$

- reparametrization path integral may describe exact effective action of IIB open superstring in $AdS_5 \times S^5$ like in flat space

- the results are applicable to the Schwinger effect in $\mathcal{N} = 4$ SYM
  – shifting of the critical electric field at strong coupling

- another potential application: polygonal light-like Wilson loops ($=$ scattering amplitudes in $\mathcal{N} = 4$ SYM)

- reparametrization path integral is crucial for consistency of off-shell string both in flat and AdS space