Polarization of gravitational wave from a point mass in a Keplerian orbit

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The elliptic orbit is described by the equations

$$x_1 = a(\cos\xi - e), \quad x_2 = a\sqrt{1-e^2}\sin\xi, \quad \tau = w_0 t = (\xi - e\sin\xi) \tag{1}$$

cf. §70 and Problems in §110 in [1] and §15 in 2. Here

$$w_0 = \frac{2\pi}{T} = \sqrt{\frac{Gm_1 m_2}{\mu a^3}}, \quad \mu = \frac{m_1 m_2}{m_1 + m(2)}$$

From here

$$\frac{d\xi}{dt} = \dot{\xi} = \frac{w_0}{1 - e\cos\xi}, \quad dt = \frac{1-\cos\xi}{w_0} d\xi \qquad (3).$$

$$\dot{x}_1 = -\frac{w_0 a \sin\xi}{1 - e\cos\xi}, \ddot{x}_1 = -w_0^2 a \frac{\cos\xi - e}{(1 - e\cos\xi)^3}, \tag{4}$$

$$\dot{x}_2 = aw_0\sqrt{1-e^2}\frac{\cos\xi}{1-e\cos\xi}, \quad {}_2 = -w_0^2 a\sqrt{1-e^2}\frac{\sin\xi}{(1-e\cos\xi)^3}. \tag{5}$$

We note the useful relations

$$x_{12} = \ddot{x}_1 x_2, \quad \frac{d^2}{dt^2}(x_i x_j) = 2(\dot{x}_{ij} + x_i\ddot{x}_j), quad i, j = 1, 2. \tag{6}$$

The energy-momentum of the system is

$$T_{ij} = \mu(\dot{x}_i\dot{x}_j + x_i\ddot{x}_j) = \frac{\mu}{2}\frac{d^2}{dt^2}(x_i x_j). \tag{7}$$

So , to obtain the Fourier transform of T_{ij}, we need the Fourier transform of $x_i x_j$:

$$[T_{ij}]_n = -\frac{\mu}{2}(w_0 n)^2[x_i x_j]_n. \tag{8}$$

Here

$$[x_i x_j]_n = \frac{1}{T}\int_0^T e^{inw_0 t} dt = \frac{1}{2\pi}\int_0^{2\pi} x_i x_j e^{inf(\xi)}(1 - e\sin\xi),$$

$$f = (1 - e\sin\xi)d\xi. \qquad (9)$$

Using x_1 in (1), we find ($z = ne$)

$$[x_1^2]_n = \frac{a^2}{2\pi}\int_0^{2\pi} e^{inf(\xi)}(1 - e\sin\xi)(\cos\xi - e)^2 =$$

$$2a^2[-\frac{1}{z^2}J_n(z) + \frac{1}{z}(1 - e^2)J_n'()z]. \qquad (10)$$

Similarly we get

$$[x_2^2]_n = 2a^2(1 - e^2)[\frac{1}{z^2}J_n(z) - \frac{1}{zJ_n'(z)}], \quad [x_1 x_2]_n =$$

$$-ia^2\sqrt{1 - e^2}\frac{2}{n}[(1 - \frac{n^2}{z^2})J_n(z) + \frac{1}{z}J_n'(z)]. \qquad (11)$$

These expressions give $[T_{ij}]_n$ in (8).

The angular distribution is given by the formula

$$dE_{\vec{q}} = \frac{8\pi G}{c^2} \left(\frac{1}{2} |T_+(q)|^2 + 2|T_\times(q)|^2 \right) \frac{d^3 q}{16\pi^3}, \tag{12}$$

where

$$\begin{aligned} T_+ = (\cos^2\theta \cos^2\varphi - \sin^2\varphi) T_{11} + 2 \sin\varphi \cos\varphi (1 \\ + \cos^2\theta) T_{12} + (\cos^2\theta \sin^2\theta - \cos^2\varphi) T_{22}, \\ T_\times = \cos\theta [\sin\varphi \cos\varphi (T_{22} - T_{11}) + (\cos^2\varphi - sin^2\varphi) T_{12}]. \end{aligned} \tag{13}$$

We note that $T_\times = 0$ for $\cos\theta = 0$ i.e. when the gravitational wave is trevelling in the direction perpendicular to the orbit plane. It is remarcable that T_+ in this case still depends on the azimuth angle φ..This dependence can be observed near third axis.

The distribution over harmonics (over n) gives information on the value of eccentrisity e. For $e = 0$ only $n = 2$ contributes in the considered approximation. The larger e the wider is the distribution over n.

The integration over the angles gives the radiated power:

$$P = \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{\ldots} n^2 g \ldots \quad P = \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{\ldots} n^2 g \ldots$$

Here

$$g_+ = \left(28\frac{n^6+n^4}{z^4} - 28\frac{3n^4+n^2}{z^2}\right.$$

$$\left. +84n^2 + 11 - 28z^2 J_n^2(z) + \left(-\frac{112n^4}{z^3}\right.\right.$$

$$\left. +\frac{196n^2}{z} - 84z\right) J_n(z) J_n'(z) + \left(28\frac{n^4+n^2}{z^2} - 56n^2 - 28 + 28z^2\right) J_n'^2(z),$$

$$g_\times = \left(4\frac{n^6+n^4}{z^4} - \frac{12n^4+4n^2}{z^2} + 12n^2 + 1 - 4z^2\right) J_n^2(z) + \left(-\frac{16n^4}{z^3}\right.$$

$$\left. +\frac{28n^2}{z} - 12z\right) J_n(z) J_n'(z) + \left(4\frac{n^4+n^2}{z^2} - 8n^4 - 4 + 4z^2\right) J_n'^2(z),$$

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