Ginzburg Conference on Physics (Lebedev Physical Institute, 2012, May 28 – June 2)

How does unipolar induction work for a Kerr black hole?

Force-Free Degenerate Electrodynamics (FFDE)

Isao OKAMOTO
National Astronomical Observatory of Japan

Key words

- Force-free, torque-free
- Inertia-free "virtual" particles
- "3+1"- formalism of GR
- lapse function α
- Dragging of inertial frames angular frequency ω
- Unipolar induction in flat and Kerr spaces
- Double DC circuits; EMF, current lines, impedances
- EMF in inertial frames in the Kerr space
- Coupling of the field line angular frequency Ω_F with ω
- Upper null surface S_N
- Two force-free domains
- Two membrane surfaces S_{ffH}, $S_{ff\infty}$ with surface resistivity

$$\mathcal{R} = \frac{4\pi}{c} = 377\mathrm{Ohm}$$

- Surface currents and torques
- Spin-down energy flux S_{SD}
- Poynting flux S_{EM}
- Total energy flux S_E
- $(E_{||})_N$ across S_N
- Pair-creation
- Extraction of rotational energy

Key expressions

$$\mathcal{E} = \frac{1}{c}\int_{\mathrm{ACB}} \boldsymbol{v} \times \boldsymbol{B} \cdot d\boldsymbol{l} = \frac{1}{c}\int_{\mathrm{ACB}} \boldsymbol{B} \times (\boldsymbol{r} \times \boldsymbol{\Omega}) \cdot d\boldsymbol{l}.$$

Eq (63.9) in Landau et al. 1984

EMF

$$\boldsymbol{v} = \kappa \boldsymbol{B} + \Omega_{\mathrm{F}} \varpi \boldsymbol{t}.$$

Mestel 1961

Flow velocity **v** in MHD

$$\boldsymbol{E}_{\mathrm{p}} = -\frac{(\Omega_{\mathrm{F}} - \omega)}{2\pi\alpha c} \boldsymbol{\nabla}\Psi, \qquad \boldsymbol{v}_{\mathrm{F}} = \frac{1}{\alpha}(\Omega_{\mathrm{F}} - \omega)\varpi \boldsymbol{t}.$$

Thorne et al. 1986

Ω_F field line angular frequency, $\mathbf{v}_F$ field line angular velocity

"3+1"-Formalism α lapse function

ω angular frequency of frame-dragging

1. Force-free degenerate electrodynamics

- We use force-free (hence torque-free) and frozen-in approximations: $\rho_e \boldsymbol{E}_p + (1/c)\boldsymbol{j} \times \boldsymbol{B} \approx 0$
- Force-freeness combines with frozen-inness to produce two-fold degeneracy; $E_\parallel = j_\perp = 0$
- Structure of a FFDE magnetosphere consists of

+ $\boxed{\text{the surface or point with unipolar inductor at work}}$ ➠ EMF

+ $\boxed{\text{force- and torque-free domains}}$ ➠ wires

+ $\boxed{\text{resistive membranes.}}$ $\mathcal{R} = \dfrac{4\pi}{c} = 377\,\mathrm{Ohm}$ ➠ impedance of a vacuum

➠ **DC circuit model**

It is "virtual" massless particles with ± charges that fill the force-free magnetosphere.

2. Unipolar Induction for pulsars

Landau & Lifshitz Course of Theoretical Physics, Vol. 8
“Electrodynamics of Continuous Media”, p. 220-1

static magnetic field **B** due to a fixed magnet. We neglect the distortion of the field by the wire itself. According to formula (63.3), the e.m.f. between the ends of the wire is

$$\mathscr{E} = \frac{1}{c}\int_{ACB} \mathbf{v}\times\mathbf{B}\cdot d\mathbf{l} = \frac{1}{c}\int_{ACB} \mathbf{B}\times(\mathbf{r}\times\boldsymbol{\Omega})\cdot d\mathbf{l}, \qquad (63.9)$$

taken along the wire. This is the required solution.

A perfectly conducting sphere, rotating with Ω about the direction of magnetization M

$\mathbf{v}=\kappa\mathbf{B}+\Omega_F\, \mathbf{t}$
(Mestel 1961)

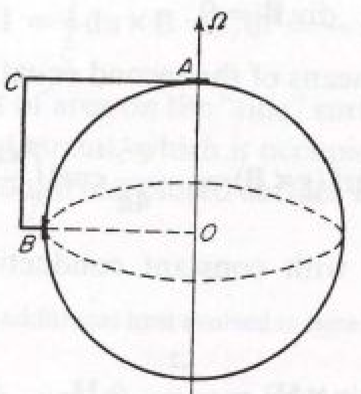


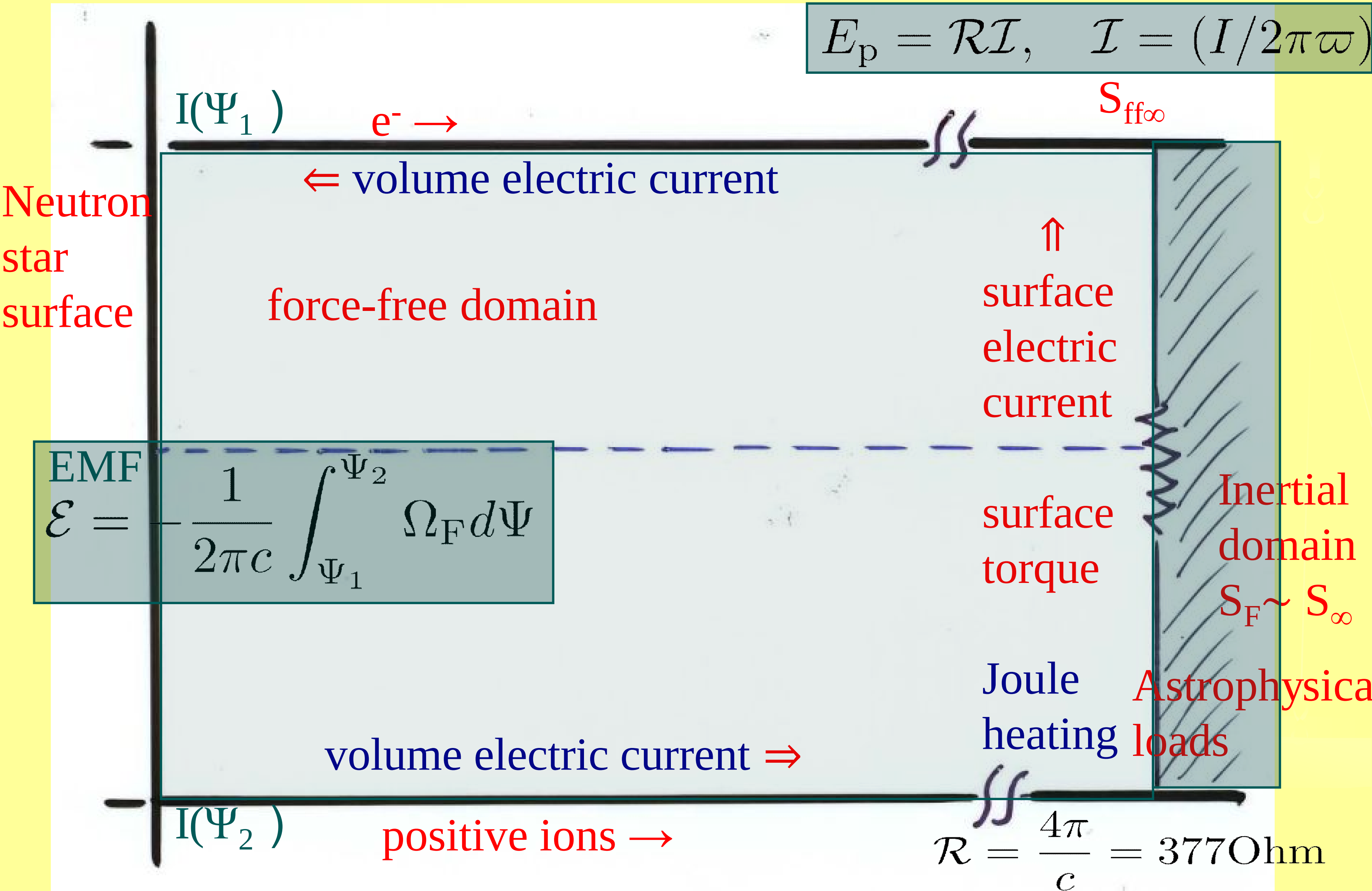
Fig. 39

OACBO=0
ACB=AOB
=OB,
OA=0

$$\mathcal{E} = -\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2} \Omega_F\, d\Psi$$

between ψ_1 and ψ_2 emanating from and pinned down at the neutron star surface
in MHD / FFDE

Pulsar DC circuit model



3. Magnetic field, electric field, and particle velocity

$$\boldsymbol{B}_{\mathrm{p}} = -\frac{1}{2\pi\varpi}(\boldsymbol{t} \times \nabla\Psi), \qquad B_{\mathrm{t}} = -\frac{2I}{\varpi\alpha c},$$

where $\Psi = \text{constant} \Rightarrow$ "field-streamline"

$I = \text{constant} \Rightarrow$ "current line"

Perfect conductivity and induction equation yield (Thorne et al. 1986);

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}, \qquad \nabla \times \boldsymbol{E} = 0$$

$$\Omega_{\mathrm{F}} = \Omega_{\mathrm{F}}(\Psi), \quad \boldsymbol{E}_{\mathrm{p}} = -\frac{(\Omega_{\mathrm{F}} - \omega)}{2\pi\alpha c}\nabla\Psi$$

Plasma motion v regulated by magnetic field B (Mestel 1961)

$$\boldsymbol{v}_{\mathrm{p}} = \kappa \boldsymbol{B}_{\mathrm{p}}, \qquad v_{\mathrm{t}} = \kappa B_{\mathrm{t}} + v_{\mathrm{F}},$$

or jointly $\quad \boldsymbol{v} = \kappa \boldsymbol{B} + \boldsymbol{v}_{\mathrm{F}}$

Magnetic slingshot

Measured by fiducial observers living in the inertial frames with ω

$$\boldsymbol{v}_{\mathrm{F}} = \text{rotational velocity of field lines}$$

$$= \frac{(\Omega_{\mathrm{F}} - \omega)\varpi}{\alpha}\boldsymbol{t}$$

$$= \begin{cases} -\infty & \rightarrow \mathrm{S}_{\mathrm{ffH}}, \\ 0 & \text{on } \mathrm{S}_{\mathrm{N}}, \\ +\infty & \rightarrow \mathrm{S}_{\mathrm{ff}\infty} \end{cases}$$

⇒ingoing wind

There must be particle source

⇒ outgoing wind

$\Rightarrow$ Magnetic slingshot

$\Rightarrow$ magnetocentrifugal winds

Electric current and particle velocity

I=I(Ψ)

GR effect

$\Downarrow$

$$\boldsymbol{j}_{\mathrm{p}} = \frac{1}{2\pi\alpha\varpi}(\boldsymbol{t}\times\nabla I) = -\frac{1}{\alpha}\frac{dI}{d\Psi}\boldsymbol{B}_{\mathrm{p}},$$

$$j_{\mathrm{t}} = -\frac{1}{8\pi^2}\left[\frac{\varpi c}{\alpha}\nabla\cdot\left(\frac{\alpha}{\varpi^2}\nabla\Psi\right) + \boxed{\frac{\varpi(\Omega_{\mathrm{F}}-\omega)}{\alpha^2 c}\nabla\Psi\cdot\nabla\omega}\right].$$

The role of "massless" particles in the force-free domains is just to carry charges

$$\boldsymbol{v} = \frac{\boldsymbol{j}}{\varrho_{\mathrm{e}}}$$

$$\boldsymbol{v}_{\mathrm{p}} = \frac{\boldsymbol{j}_{\mathrm{p}}}{\varrho_{\mathrm{e}}} = \frac{(\boldsymbol{t}\times\nabla I)}{2\pi\varpi\alpha\varrho_{\mathrm{e}}} = -\frac{1}{\alpha\varrho_{\mathrm{e}}}\frac{dI}{d\Psi}\boldsymbol{B}_{\mathrm{p}}$$

$$v_{\mathrm{t}} = \frac{j_{\mathrm{t}}}{\varrho_{\mathrm{e}}}$$

4. Black hole unipolar induction

- How and where is the field line angular frequency Ω_F determined in terms of the hole's angular frequency Ω_H? Where are they pinned down ?
- There is no material at the horizon by which to anchor field lines, i.e. $\Omega_F \neq \Omega_H$.
- The key to solve this questions; $\Leftrightarrow E_p$
  - α, the red-shift factor/lapse function
  - ω, angular frequency of frame-dragging

$$\boldsymbol{E}_{\mathrm{p}} = -\frac{(\Omega_{\mathrm{F}} - \omega)}{2\pi\alpha c}\nabla\Psi$$

"3+1"-formalism in Boyer-Lindquist coordinates for GR

Physical meanings of Ω_F

$$\Omega_{\rm F} = \text{angular frequency field lines} \quad (1)$$

$$= -\frac{d\phi}{d\Psi} = \text{potential gradient} \quad (2)$$

$$\Rightarrow \text{Unipolar inductor}$$

$$\Rightarrow \text{Electromotive Force}$$

In order to fix $\Omega_F(\Psi)$ and produce EMFs, magnetic fluxes must be pinned down on plasma particles.

From $$\boldsymbol{E}_{\rm p} = -\frac{(\Omega_{\rm F} - \omega)}{2\pi\alpha c}\nabla\Psi,$$

the potential difference:

5. EMF for double DC circuits

$$\text{Potential Difference} = -\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2}[\Omega_F(\Psi) - \omega(\Psi,\ell)]d\Psi$$

$$\mathcal{E}_{\mathrm{ffH}} = +\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2}[\Omega_H - \Omega_F(\Psi)]d\Psi \qquad ;\text{on } S_{\mathrm{ffH}} \text{ at } \ell = \ell_H$$

$\omega=\Omega_H$

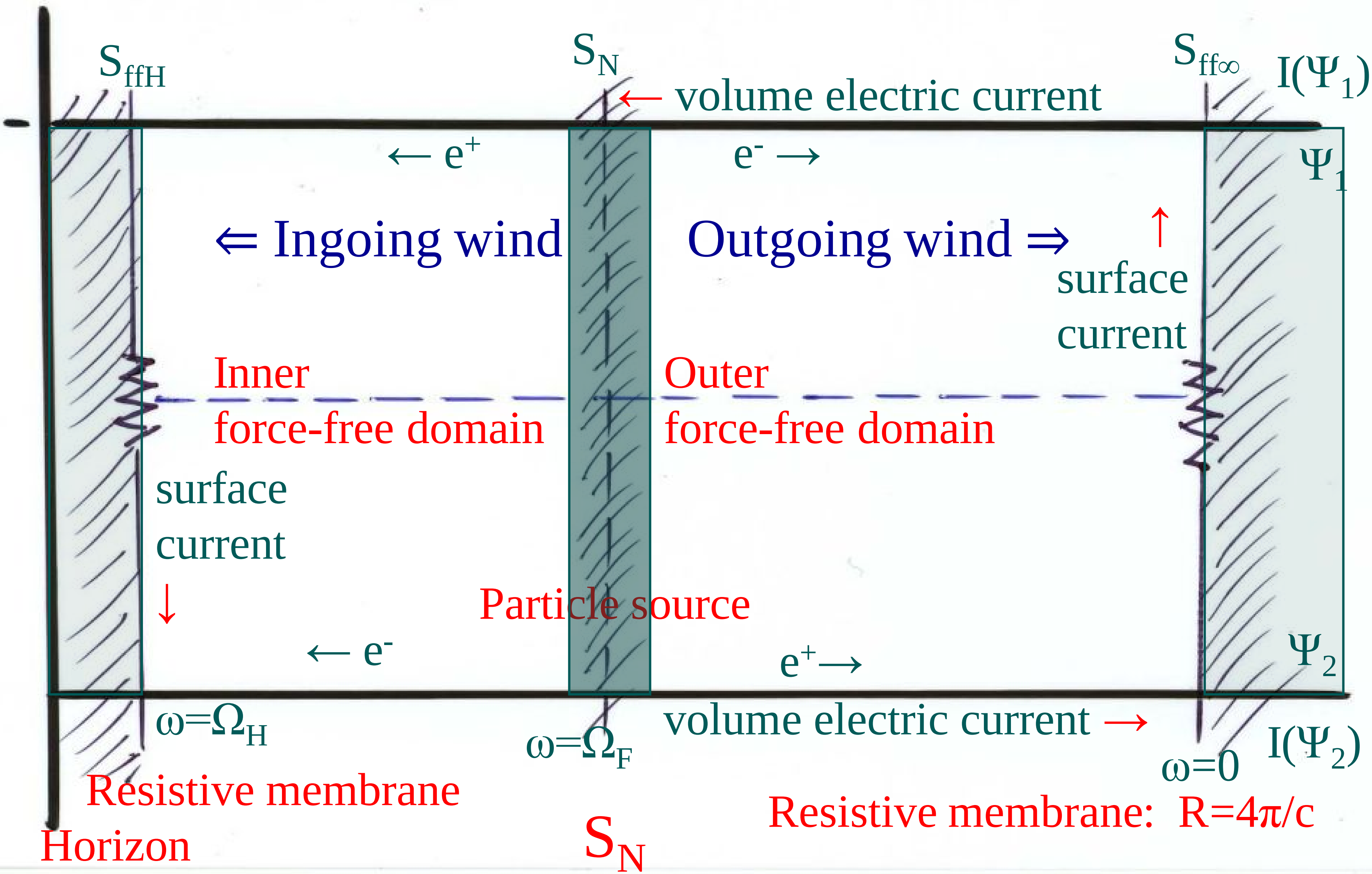
$$\mathcal{E}_{\mathrm{ff}\infty} = -\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2}\Omega_F(\Psi)d\Psi \qquad ;\text{on } S_{\mathrm{ff}\infty} \text{ at } \ell = \ell_\infty$$

$\omega=0$

on the two resistive membranes
terminating force-free domains

DC circuit =
EMF
\+ Current lines in Force-free domains
\+ Impedances on the resistive membranes
with surface currents ≡ Astrophysical loads

An Image of Double DC circuits in FFDE



6. Problems to solve

- The double-eigenvalue problem: for a given structure

  the criticality condition for $I(\Psi)$'s at $S_{ffH}/S_{ff\infty}$

  the boundary condition for $\Omega_F(\Psi)$ at S_N,

separable, but coupled. ⇕ iterations

- Grad-Shafronov equation

  for the field structure Ψ in the force-free domains,

  nonlinear containing $I(\Psi)$'s and $\Omega_F(\Psi)$

  elliptic 2nd-order partial differential eq.

7. Eigen-functions, $I(\Psi)$'s and $\Omega_F(\Psi)$

Criticality condition for ingoing/outgoing winds Ohm's law

$$I(\Psi) = \begin{cases} \frac{1}{2}(\Omega_{\rm H} - \Omega_{\rm F})(B_{\rm p}\varpi^2)_{\rm ffH} \equiv I_{\rm in}, & \text{at } \mathrm{S_{iF}} \approx \mathrm{S_{ffH}}, \quad \omega \approx \Omega_{\rm H}, \\ \frac{1}{2}\Omega_{\rm F}(B_{\rm p}\varpi^2)_{\rm ff\infty} \equiv I_{\rm out}, & \text{at } \mathrm{S_{oF}} \approx \mathrm{S_{ff\infty}}, \quad \omega \approx 0. \end{cases}$$

Boundary condition $I_{\rm in} = I_{\rm out}$ **at** $\mathrm{S_N}$, $\omega = \Omega_{\rm F}$

$$\Omega_{\rm F} = \frac{\Omega_{\rm H}}{1+\zeta}, \quad \zeta \equiv \frac{(B_{\rm p}\varpi^2)_{\rm ff\infty}}{(B_{\rm p}\varpi^2)_{\rm ffH}}.$$

⇕

"Continuity of energy and angular momentum flux at S_N between the inner and outer domains"

Non-FFDE plasma processes are hidden under S_N, of pair-creation and pinning down magnetic field lines on particles, so that $\Omega_F=\omega(\mathrm{ell}_N)$

8. Grad-Shafronov Equation for Ψ in the "force-free" domains

- Non-linear, in that it contains two unknown functions $I(\Psi)$ and $\Omega_F(\Psi)$
- 2^{nd}-order partial differential equation
- Elliptic in the force-free domain

$$\nabla \cdot \left[\frac{\alpha}{\varpi^2} \left(1 - \frac{\varpi^2 (\Omega_F - \omega)^2}{\alpha^2 c^2} \right) \nabla \Psi \right] + \frac{(\Omega_F - \omega)}{\alpha c^2} \frac{d\Omega_F}{d\Psi} |\nabla \Psi|^2 + \frac{8\pi^2}{\alpha \varpi^2 c^2} \frac{dI^2}{d\Psi} = 0$$

In flat space, with $\alpha = 1$ and $\omega = 0$, "pulsar equation". Split-monopole $\Psi = \Psi_0 (1 - \cos\theta)$ is an exact solution for Ω_F=const, $I = B_p \varpi^2 / 2$.

Exact solution of GS equation in the slow-rotation limit for a split-monopole

$h=a/r_H << 1$
perturbation

$$\Psi = \Psi_0(1 - \cos\theta + \boxed{h^2} f(r)\sin^2\theta\cos\theta),$$

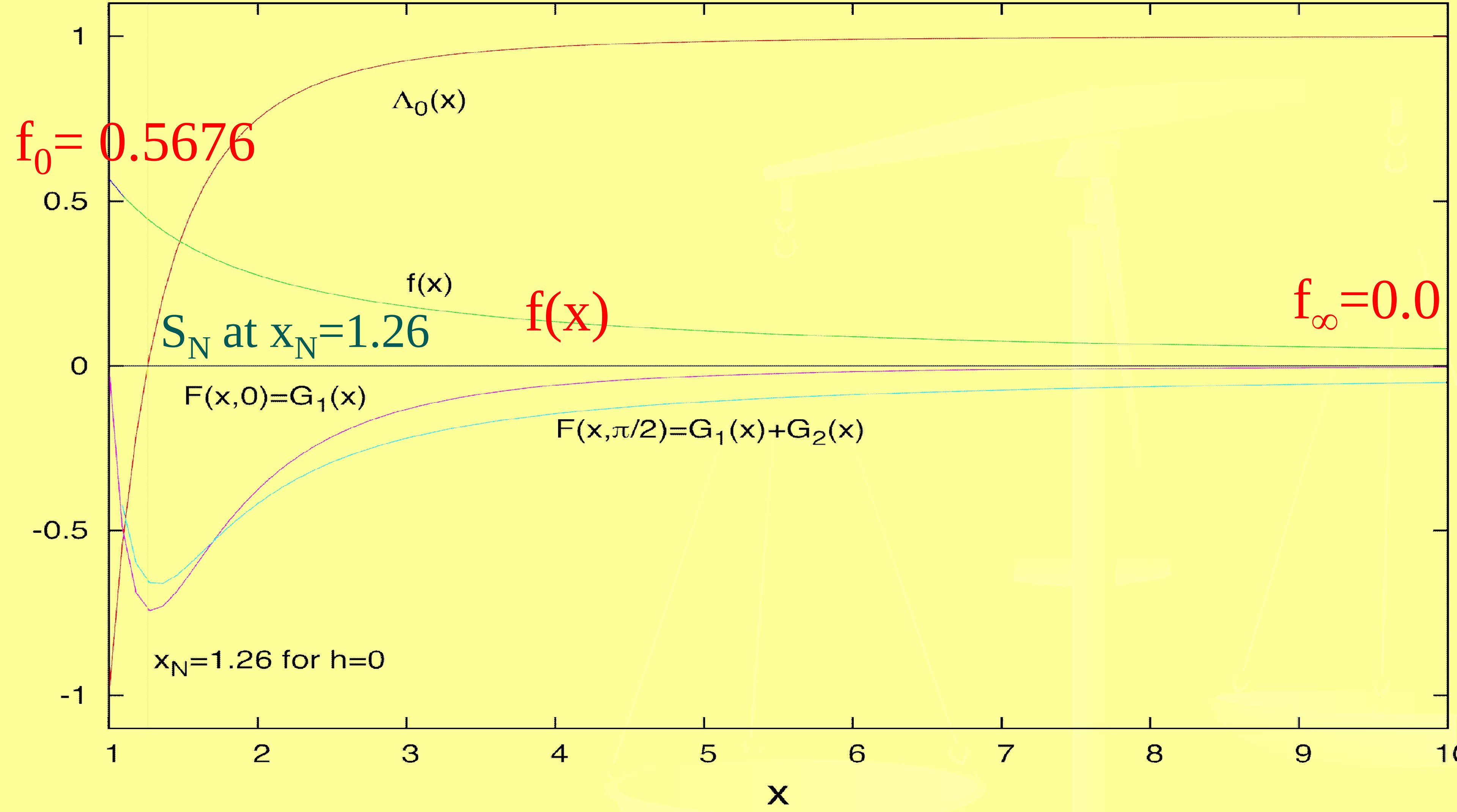
$$\boxed{x(x-1)f'' + f' - 6f = -\frac{2}{x}\left(1+\frac{1}{x}\right)}$$

$$\boxed{\begin{aligned} f(x) = 8x^3 \Bigg\{ \left(1-\frac{3}{4x}\right) \Bigg[& I_{\mathrm{A}}(x) - \ln\left(1-\frac{1}{x}\right)\ln x \\ & -\frac{1}{x}\left(1+\frac{1}{4x}+\frac{1}{9x^2}\right)\Bigg] - \frac{\ln x}{x}\left(1-\frac{1}{4x}-\frac{1}{24x^2}\right)\Bigg\}, \end{aligned}}$$

$$I_{\mathrm{A}}(x) = \sum_{n=1}^{\infty}\frac{1}{n^2x^n} = -\int_0^1 \frac{1}{t}\ln\left(1-\frac{t}{x}\right)dt.$$

(see Blandford-Znajek 1976; Okamoto 2009)

Split-monopolar exact solution in the slow-rotation limit



9. Electromagnetic and total energy fluxes

Electromagnetic energy flux

$$\boldsymbol{S}_{\mathbf{EM}} = \frac{\alpha c}{4\pi}(\boldsymbol{E} \times \boldsymbol{B}) = \frac{(\Omega_{\mathrm{F}} - \omega)I}{2\pi\alpha c}\boldsymbol{B}_{\mathbf{p}}$$

$$= \begin{cases} > 0 & ;\text{outward for } \ \omega < \Omega_{\mathrm{F}}, \\ < 0 & ;\text{inward for } \ \omega > \Omega_{\mathrm{F}}. \end{cases}$$

dependent on ω

Total energy flux

$$\boldsymbol{S}_{\mathbf{E}} = \boldsymbol{S}_{\mathbf{EM}} + \boldsymbol{S}_{\mathbf{SD}} = \frac{\Omega_{\mathrm{F}} I}{2\pi\alpha c}\boldsymbol{B}_{\mathbf{p}}$$

independent on ω

Surface torque and angular momentum flux

$$\frac{dJ}{dt} = -\oint (\alpha \boldsymbol{\mathcal{I}}_{\mathbf{ffH}}/c \times \boldsymbol{B}_{\mathbf{p}}) \cdot \varpi \boldsymbol{t}) dA$$

Angular momentum loss of the hole by the surface torque

$$= -\frac{1}{2\pi c}\oint I(\Psi) d\Psi = -\oint \alpha \boldsymbol{S}_{\mathbf{J}} \cdot d\boldsymbol{A}$$

Angular momentum flux
independent of ω

$$\boxed{\boldsymbol{S}_{\mathbf{J}} = \frac{I(\Psi)}{2\pi\alpha c}\boldsymbol{B}_{\mathbf{p}}}$$

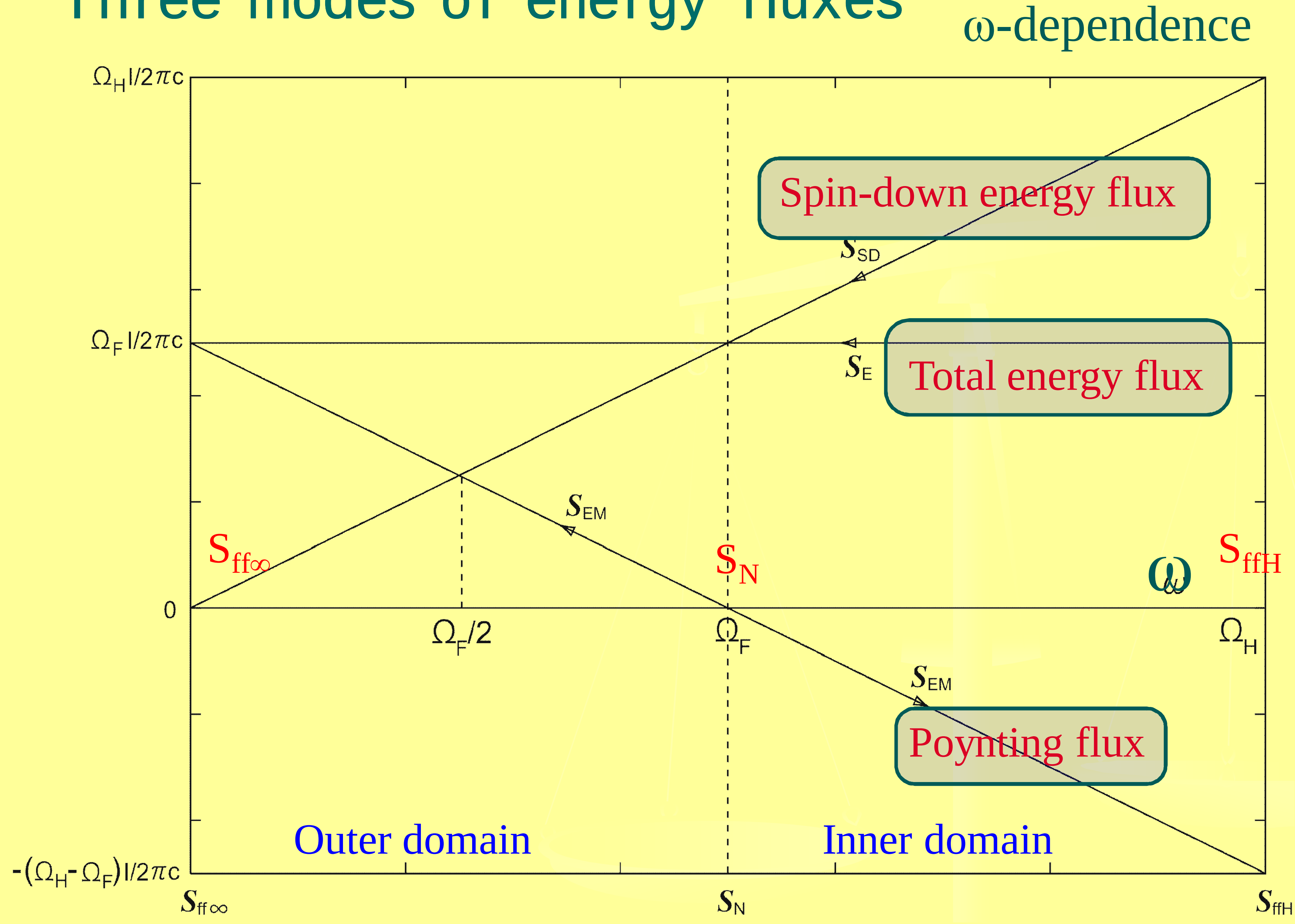
Spin-down energy flux of
purely general-relativistic origin
Dependent on ω

$$\boxed{\boldsymbol{S}_{\mathbf{SD}} = \omega \boldsymbol{S}_{\mathbf{J}} = \frac{\omega I}{2\pi\alpha c}\boldsymbol{B}_{\mathbf{p}}}$$

$$\oint \alpha \boldsymbol{S}_{\mathbf{SD}} \cdot d\boldsymbol{A}\Big|_{\mathrm{ffH}} = \frac{\Omega_{\mathrm{H}}}{2\pi c}\oint I(\Psi) d\Psi = -\Omega_{\mathrm{H}}\frac{dJ}{dt}$$

Loss of the hole's rotational energy

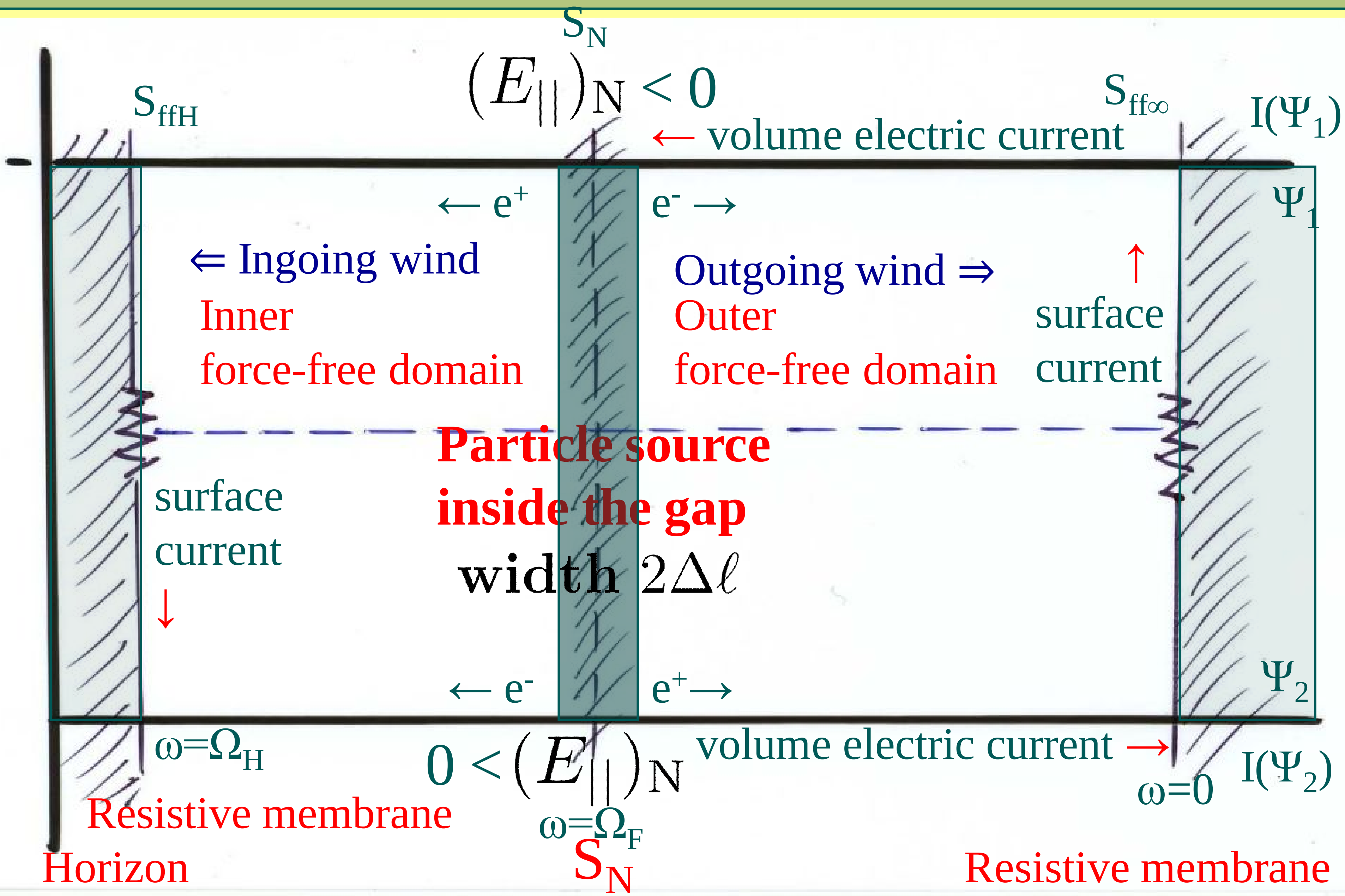
Three modes of energy fluxes



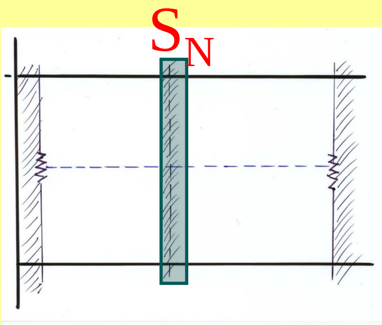
10. Physical roles of resistive membranes

- S_{ffH} : inertial domain from S_{iF} to S_H where Joule dissipation leads to entropy increase
- $S_{ff\infty}$: inertial domain from S_{oF} to S_∞ where astrophysical loads is existent in MHD
- Surface currents on S_{ffH} ➡ surface torque ➡ extraction of the hole's angular momentum and rotational energy
- Criticality condition ➡ Ohm's law
- S_{ffH} : Joule dissipation ➡ entropy increase
- $S_{ff\infty}$: Joule dissipation ➡ MHD acceleration ➡ jets

11. Particle source inside the gap under S_N



$(E_{||})_N$ across the gap under S_N



PDin PDout

$$\omega(\ell) \approx \omega(\ell_{\rm N}) \pm \Delta\omega,$$

$$\omega(\ell_{\rm N}) = \Omega_{\rm F}, \quad \Delta\omega = |(\partial\omega/\partial\ell)|_{\rm N}\Delta\ell$$

$$\text{at } \ell = \ell_{\rm N} \mp \Delta\ell \equiv \bar{\ell}_{\rm in}/\bar{\ell}_{\rm out}$$

$$\mathrm{PD} = -\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2} [\Omega_{\rm F}(\Psi) - \omega(\Psi, \ell)]d\Psi$$

$$\mathrm{PD}_{\rm in} \text{ or } \mathrm{PD}_{\rm out} = \pm\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2} \Delta\omega d\Psi,$$

$$(E_{||})_{\rm N} \approx \mp\frac{|\mathrm{PD}_{\rm in}| + |\mathrm{PD}_{\rm out}|}{2\Delta\ell} \approx \mp\frac{1}{2\pi c}\int_{\Psi_1}^{\Psi_2} \left|\frac{\partial \ln\omega}{\partial\ell}\right|_{\rm N} \Omega_{\rm F} d\Psi$$

- Pair creation and charge-separation inside the gap
- Inflow and outflow of massless charges in the force-free domains

Simple image of non-FFDE processes under S_N

Phenomenological analysis

Plasma source at S_N with $\omega \approx \Omega_F$

Pinning down magnetic field lines at plasma. Fixing $\Omega_F = \omega(I_N)$ and S_N

unipolar induction

Potential difference between field lines Ψ_1 and Ψ_2 chosen as $I(\Psi_1)=I(\Psi_2)$

$(E_{||})_N$ local inside the gap across S_N

pair creation

12a. Summary

- A classical process of unipolar induction will produce EMFs strong enough to make a black hole magnetosphere active.
- We have to elucidate some non-FFDE processes under S_N beyond conjecture

(i) How to pin field lines down on plasma particles near S_N and to determine $\Omega_{\rm F} = \omega(\ell_{\rm N}) \approx (\Omega_{\rm H}/2)$

(ii) How $(E_{||})_N$ create pair-particles inside the non-FFDE gap at S_N

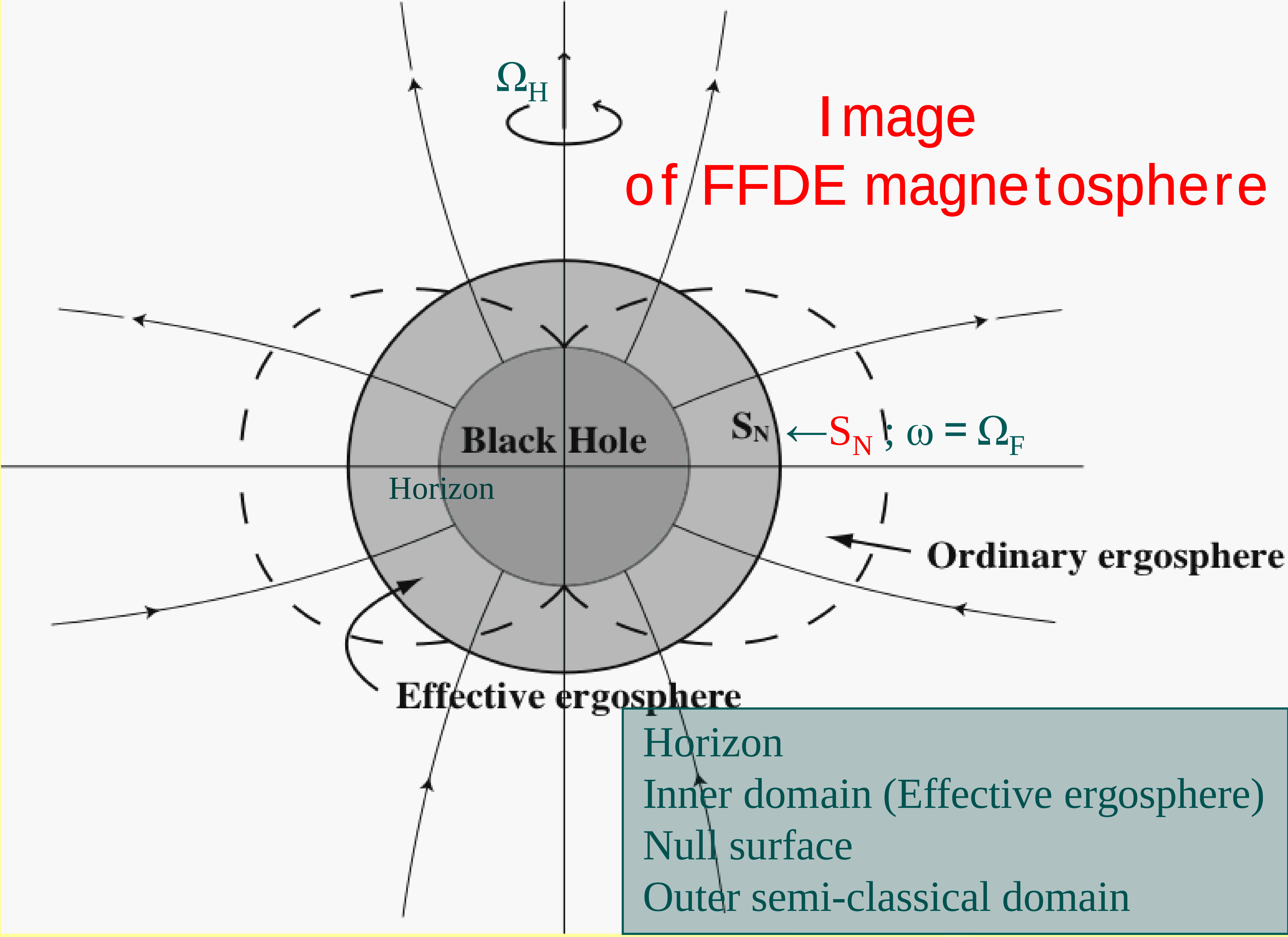
(iii) etc.

Also, beyond FFDE, construct MHD DC circuit model for B. H. magnetospheres

11b. Summary

- Landau et al.'s concept for unipolar induction will be applicable to a hole magnetosphere with inertial frames dragged with ω;
- The angular frequency of field lines Ω_F couples with ω, to create the inner ($\Omega_H \geq \omega \geq \Omega_F$) and outer domains ($\Omega_F \geq \omega \geq 0$), with the interface S_N at $\omega = \Omega_F$.
- Ω_F will be determined as the eigenvalue of this general-relativistic system due to the criticality-boundary conditions in the steady axisymmetric state.
- GS equation for field structure must be solved, together with the eigenvalue Ω_F.
- Field lines will be anchored at the plasma source by pair creation at work at the inetrface S_N, $\Omega_F = \omega(\mathrm{ell}_N)$.
- Dual DC circuit model is useful with EMF's, current lines and impedances for a Kerr hole magnetosphere.

Image of FFDE magnetosphere



Horizon
Inner domain (Effective ergosphere)
Null surface
Outer semi-classical domain

13. Remaining questions

- Microphysics inside the gap hidden under S_N:

(i) Plasma supply by pair-creation,

(ii) Process of pinning down of magnetic field lines onto plasma, and determining $\Omega_F \approx \omega(\mathrm{ell}_N)$

- Find exact solutions of GS equation !?

Exact solution is impossible to find except BZ solution ?

- Beyond FFDE, construct a MHD model of magnetosphere, with astrophysical loads such as gamma ray jets, etc.

14. References

- Blandford, R. & Znajek, R.L., 1977, MNRAS, 179, 465
- Goldreich, P. & Julian, W.H., ApJ, 157, 869
- Landau, L.D. et al., 1984, Electrodynamics of Continuous Media, 2nd Ed., Elsvier, Tokyo
- Macdonald, D.A. & Thorne, K., MNRAS, 198, 345
- Mestel, L., 1961, MNRAS, 122, 473
- Michel, F.C., 1973, ApJ, 180, L133
- Okamoto, I., 1974, MNRAS, 167, 457
- Okamoto, I., 1992, MNRAS, 254, 192
- Okamoto, I., 2006, PASJ, 58, 1047
- Okamoto, I., 2009, PASJ, 61, 971
- Okamoto, I., 2012, PASJ, 64, 50 (June 25)
- Thorne, et al., 1986, Black holes: The Membrane Paradigm, Yale University Press
- Znajek, R.L., 1977, MNRAS, 179, 457