ФЕДЕРАЛЬНОЕ ГОСУДАРСТВЕННОЕ БЮДЖЕТНОЕ УЧРЕЖДЕНИЕ НАУКИ
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ФИАН

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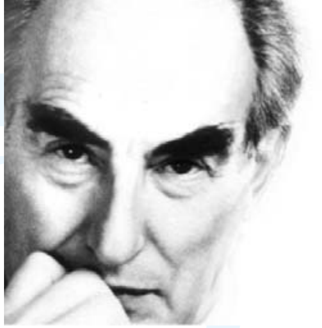

Spatial-Temporal Patterns Arising in Active Media in the Vicinity of the Wave Bifurcation

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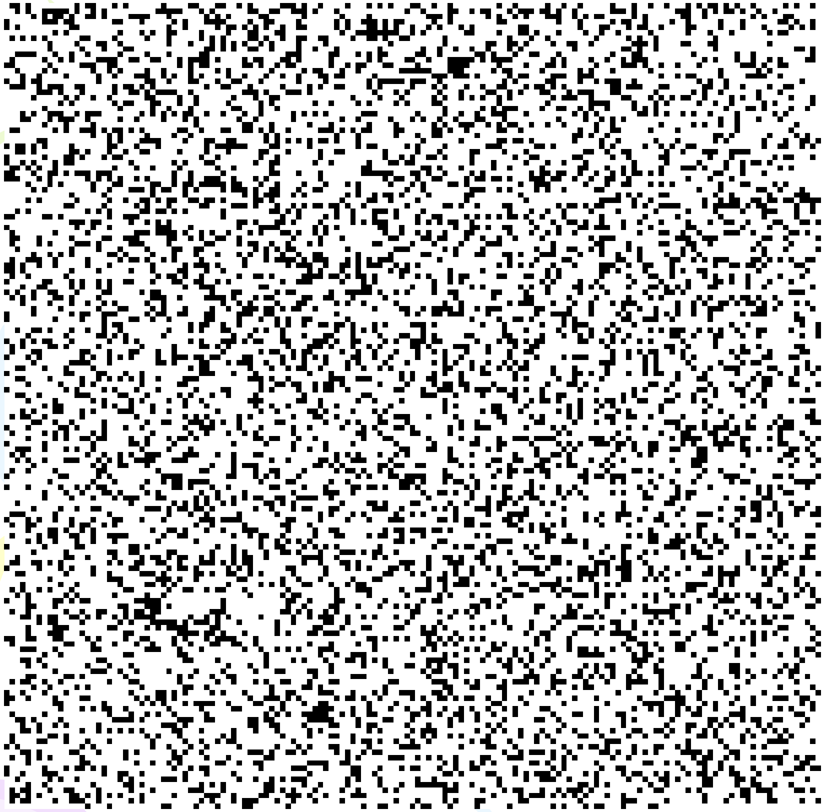
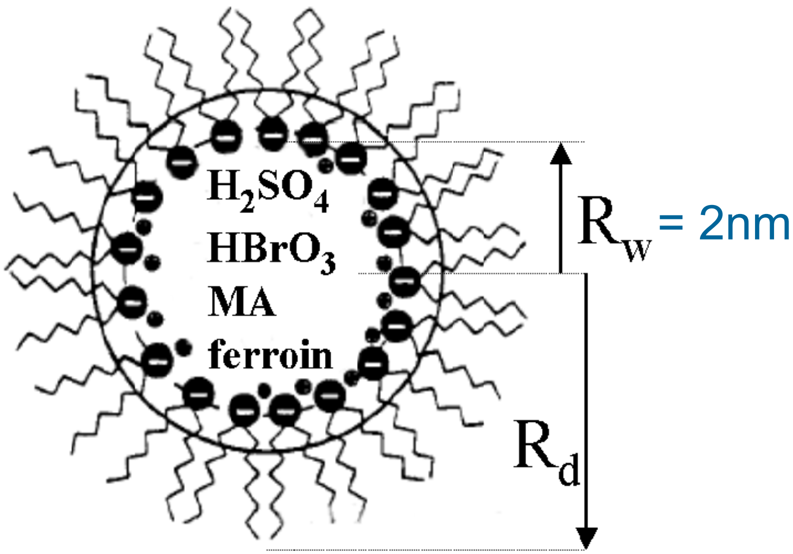
Talk outline

- Experimental background
- Diffusion instability
- Linear analysis of a three-variable system
- Spatial-temporal patterns in a multidimensional active medium caused by polymodal interaction near the wave bifurcation
- Mechanism of switching from standing to traveling waves accompanied by halving of the wavelength
- Conclusion

Microemulsion of water droplets, containing BZ reagents, in oil



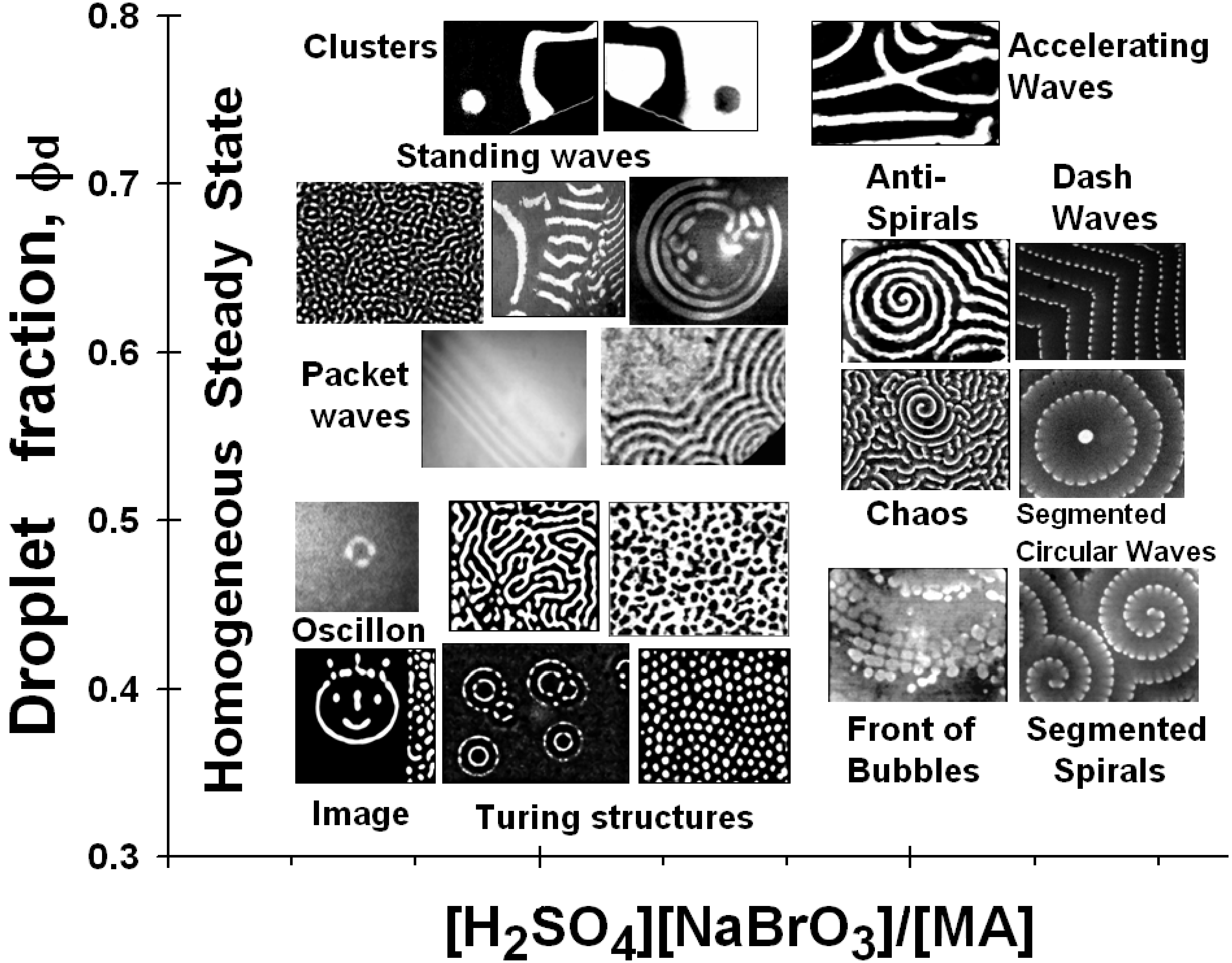
(Vanag et al.)



$1\ \mu m \times 1\ \mu m$

Volume fraction of droplets $\varphi_d = 0.3$

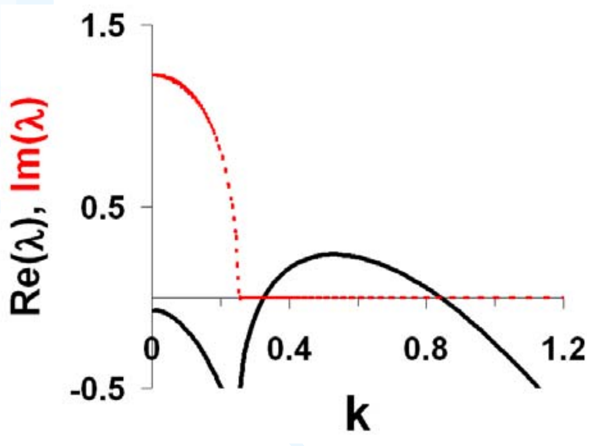
Variety of patterns in BZ-AOT microemulsion
(Vanag, 2004)



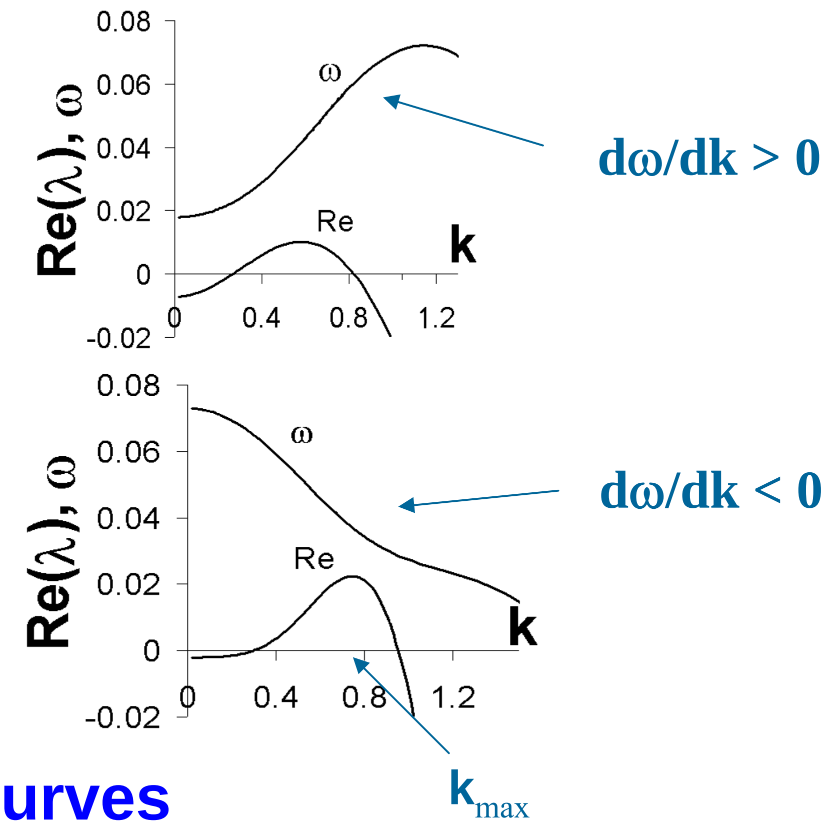
Diffusion instability

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u},\ldots) + \nabla(\mathbf{D}\nabla\mathbf{u})$$

Turing instability



Wave instability



Dispersion curves

Turing instability in a two-variable system

$$\frac{\partial u}{\partial t} = au + bv + D_1 \frac{\partial^2 u}{\partial r^2},$$

$$\frac{\partial v}{\partial t} = cu + dv + D_2 \frac{\partial^2 v}{\partial r^2}$$

Uniform state (0,0) becomes unstable, if

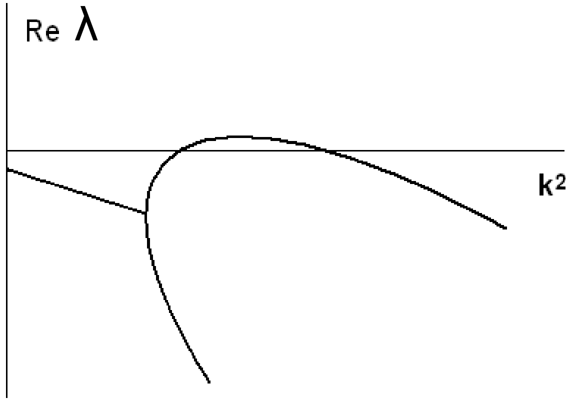
a) $ad\text{-}bc > 0$,

b) $a+d<0$,

c) $d<0$,

d) $a>0$,

e) $D_2 a + D_1 d > 2\sqrt{D_1 D_2 (ad - bc)}$



Linear analysis of a three-variable system

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v,w) + D_1 \Delta u, \\ \dfrac{\partial v}{\partial t} = g(u,v,w) + D_2 \Delta v, \\ \dfrac{\partial w}{\partial t} = h(u,v,w) + D_3 \Delta w. \end{cases}$$

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} = a_{11}\overline{u} + a_{12}\overline{v} + a_{13}\overline{w} + D_1 \Delta \overline{u}, \\ \dfrac{\partial \overline{v}}{\partial t} = a_{21}\overline{u} + a_{22}\overline{v} + a_{23}\overline{w} + D_2 \Delta \overline{v}, \\ \dfrac{\partial \overline{w}}{\partial t} = a_{31}\overline{u} + a_{32}\overline{v} + a_{33}\overline{w} + D_3 \Delta \overline{w}. \end{cases}$$

Borina , Polezhaev, 2011

Characteristic equation

$$\begin{vmatrix} a_{11}-\mathrm{k}^2 D_1-\Lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\mathrm{k}^2 D_2-\Lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\mathrm{k}^2 D_3-\Lambda \end{vmatrix} = 0$$

$$\Lambda^3 - A\Lambda^2 + B\Lambda - C = 0$$

$$A = \sigma - \boldsymbol{k}^2 (D_1 + D_2 + D_3),$$
$$B = \Sigma - \boldsymbol{k}^2 (D_1(a_{22}+a_{33}) + D_2(a_{11}+a_{33}) + D_3(a_{11}+a_{22})) + \boldsymbol{k}^4 (D_1 D_2 + D_1 D_3 + D_2 D_3),$$
$$C = \Delta - \boldsymbol{k}^2 \sum_{i=1}^{3} D_i \Theta_i + \boldsymbol{k}^4 \cdot (D_1 D_2 a_{33} + D_1 D_3 a_{22} + D_2 D_3 a_{11}) - \boldsymbol{k}^6 D_1 D_2 D_3,$$
$$\sigma = a_{11} + a_{22} + a_{33}, \quad \Sigma = \sum_{i=1}^{3} \Theta_i, \; \Theta_i = a_{jj} a_{kk} - a_{jk} a_{kj}, \; i \neq j \neq k,$$
$$\Delta = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Relations between coefficients of the cubic equation and its roots:

$$\begin{cases} A = \Lambda_1 + \Lambda_2 + \Lambda_3, \\ B = \Lambda_1\Lambda_2 + \Lambda_2\Lambda_3 + \Lambda_1\Lambda_3, \\ C = \Lambda_1\Lambda_2\Lambda_3, \\ AB - C = (\Lambda_1 + \Lambda_2)(\Lambda_2 + \Lambda_3)(\Lambda_1 + \Lambda_3). \end{cases}$$

The examined state of the system is stable if for all eigenvalues of characteristic equation $\operatorname{Re}\Lambda_i(k^2) < 0, \quad i = 1, 2, 3.$

Uniform state is stable if and only if

$$\begin{cases} A < 0, \\ B > 0, \\ C < 0, \\ AB - C < 0. \end{cases}$$

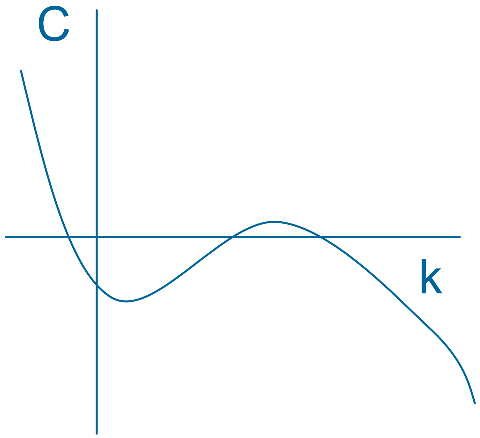
Turing bifurcation

The function $C(k^2)$ has the form $C(k^2) = \Delta - \alpha k^2 + \beta k^4 - \delta k^6$

where $\alpha = \sum_{i=1}^{3} D_i \Theta_i$ $\quad \beta = D_1 D_2 a_{33} + D_1 D_3 a_{22} + D_2 D_3 a_{11}$ $\quad \delta = D_1 D_2 D_3$

For Turing instability to take place the following condition should be met

$$C_{\max}(k_0^2) = \Delta + \frac{1}{27\delta^2}\left\{2\left(\beta^2 - 3\alpha\delta\right)^{\frac{3}{2}} + \beta\left(2\beta^2 - 9\alpha\delta\right)\right\} > 0,$$



where $k_0^2 = \frac{1}{3\delta}\left(\beta + \sqrt{\beta^2 - 3\alpha\delta}\right)$

The conditions for Turing instability in the case $a_{11} > 0, \quad D_1 << D_2, D_3$

$$\sigma < 0, \quad \Sigma > 0, \quad \Delta < 0,$$

$$\sigma \cdot \Sigma - \Delta < 0,$$

$$\frac{D_2 D_3}{D_1^2} > \frac{27}{4} \frac{(-\Delta)}{a_{11}^3}.$$

Wave instability

$$F(k^2) = AB - C = \sigma\Sigma - \Delta - \alpha k^2 + \beta k^4 - \delta k^6.$$

where

$$\alpha = D_1(\sigma^2 - a_{11}^2 - a_{12}a_{21} - a_{13}a_{31}) + D_2(\sigma^2 - a_{22}^2 - a_{12}a_{21} - a_{23}a_{32}) + D_3(\sigma^2 - a_{33}^2 - a_{13}a_{31} - a_{23}a_{32}),$$

$$\beta = (D_1 + D_3)(D_2 + D_3)(a_{11} + a_{22}) + (D_1 + D_2)(D_2 + D_3)(a_{11} + a_{33}) + (D_1 + D_2)(D_1 + D_3)(a_{22} + a_{33}),$$

$$\delta = (D_1 + D_2)(D_2 + D_3)(D_1 + D_3)$$

For wave instability to take place the following condition should be met

$$F_{\max}(k_0^2) = \sigma \cdot \Sigma - \Delta + \frac{1}{27\delta^2}\left\{2\left(\beta^2 - 3\alpha\delta\right)^{\frac{3}{2}} + \beta\left(2\beta^2 - 9\alpha\delta\right)\right\} > 0$$

where $$k_0^2 = \frac{1}{3\delta}\left(\beta + \sqrt{\beta^2 - 3\alpha\delta}\right)$$

The conditions for wave instability in the case $(a_{11}+a_{22})>0,\quad \sigma<0,\quad D_3>>D_1,D_2$

$$\Sigma>0,\quad \Delta<0,$$

$$\sigma\cdot\Sigma-\Delta<0,$$

$$\left(\frac{D_1+D_2}{D_3}\right)^2<\frac{4}{27}\frac{(a_{11}+a_{22})^3}{(\Delta-\sigma\cdot\Sigma)}.$$

$\widetilde{A}_j$

Spatial-Temporal Patterns in a Multidimensional Active Medium Caused by Polymodal Interaction Near the Wave Bifurcation

$$\partial_t \widetilde{A}_j = \widetilde{A}_j - (1 - ic_1)\widetilde{A}_j \left|\widetilde{A}_j\right|^2 - h(1 - ic_2)\widetilde{A}_j \cdot \sum_{k=1, k\neq j}^{N} \left|\widetilde{A}_k\right|^2, \quad j \in \overline{1, N}. \quad (1)$$

$\widetilde{A}_j$ are complex amplitudes of modes corresponding to equal in length but different in direction wave vectors becoming unstable due to the wave bifurcation.

$$\widetilde{A}_j = A_j e^{i\varphi_j}$$

$$\partial_t A_i = A_i - A_i^3 - A_i \cdot h \sum_{j=1, j\neq i}^{N} A_j^2, \quad i \in \overline{1, N}. \quad (2)$$

Stationary states of (2)

$$A_i^{st} = \begin{cases} \dfrac{1}{\sqrt{1+(p-1)h}}, & i \in \overline{1,p}, \\ 0, & i \in \overline{p+1,N} \end{cases} \qquad (3)$$

Linear stability of a stationary state (3)

Equations (2) literalized near the point (3)

$$\begin{cases} \delta\dot{A}_i = \dfrac{2}{1+(p-1)h}\left(-\delta A_i - h\sum\limits_{\substack{j=1,\\ j\neq i}}^{p}\delta A_j\right), & i\in\overline{1,p}, \\ \delta\dot{A}_i = \dfrac{(1-h)}{1+(p-1)h}\delta A_i, & i\in\overline{p+1,N}. \end{cases} \qquad (4)$$

Eigenvalues of the dispersion equation for the set (4)

$$\lambda_k = \begin{cases} -2(1+(p-1)h), & k=1 \\ 2(h-1), & k\in\overline{2,p} \\ 1-h, & k\in\overline{p+1,N} \end{cases}$$

Theorem

If $h \in (1, \infty)$ then equations (1) have N stable stationary states such that only one of the amplitudes is nonzero and its magnitude equals unity, while all the others are zero.

If $h \in (0, 1)$ then all the amplitudes are nonzero and have the same magnitudes equal to $1/\sqrt{1+(N-1)h}$

All other stationary points are unstable for any *h*.

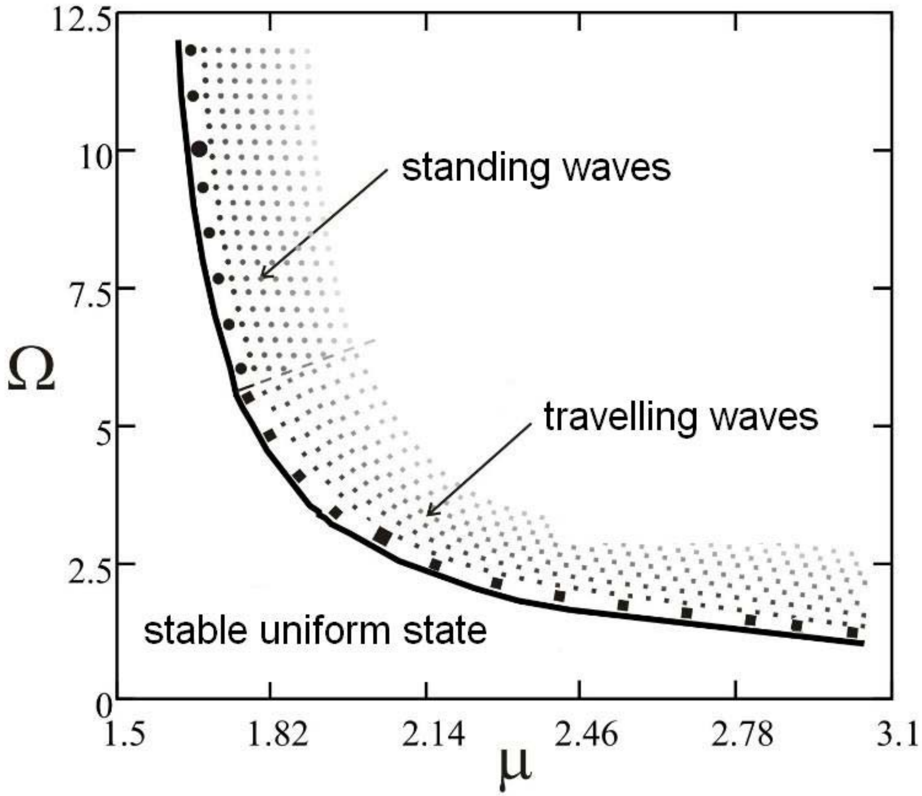
The modified Gierer-Mainhardt model:
parametric analysis and numerical simulations

$$\begin{cases} \dfrac{dX}{dt} = (\rho + \dfrac{X^2}{Y} - \mu Z - cX + dZ)\Omega + D_1 \nabla^2 X, \\ \dfrac{dY}{dt} = Z^2 - Y + D_2 \nabla^2 Y, \\ \dfrac{dZ}{dt} = cX - dZ + D_3 \nabla^2 Z \end{cases}$$

These equations have one stationary point:

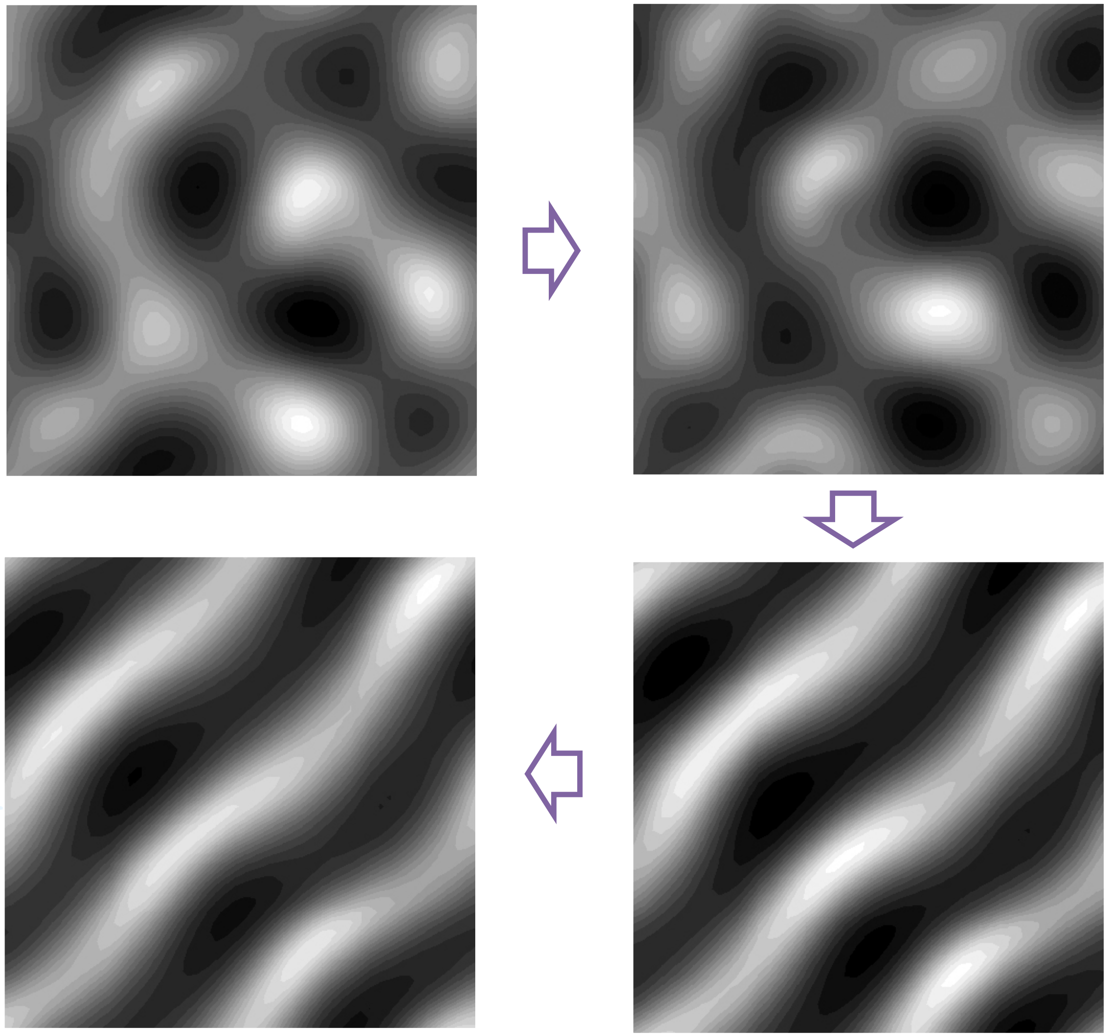
$$X_0 = \frac{\rho + 1}{\mu}, \quad Y_0 = \left(\frac{\rho + 1}{\mu}\right)^2, \quad Z_0 = \frac{c(\rho + 1)}{d\mu}$$

Parametric space of the model (5)

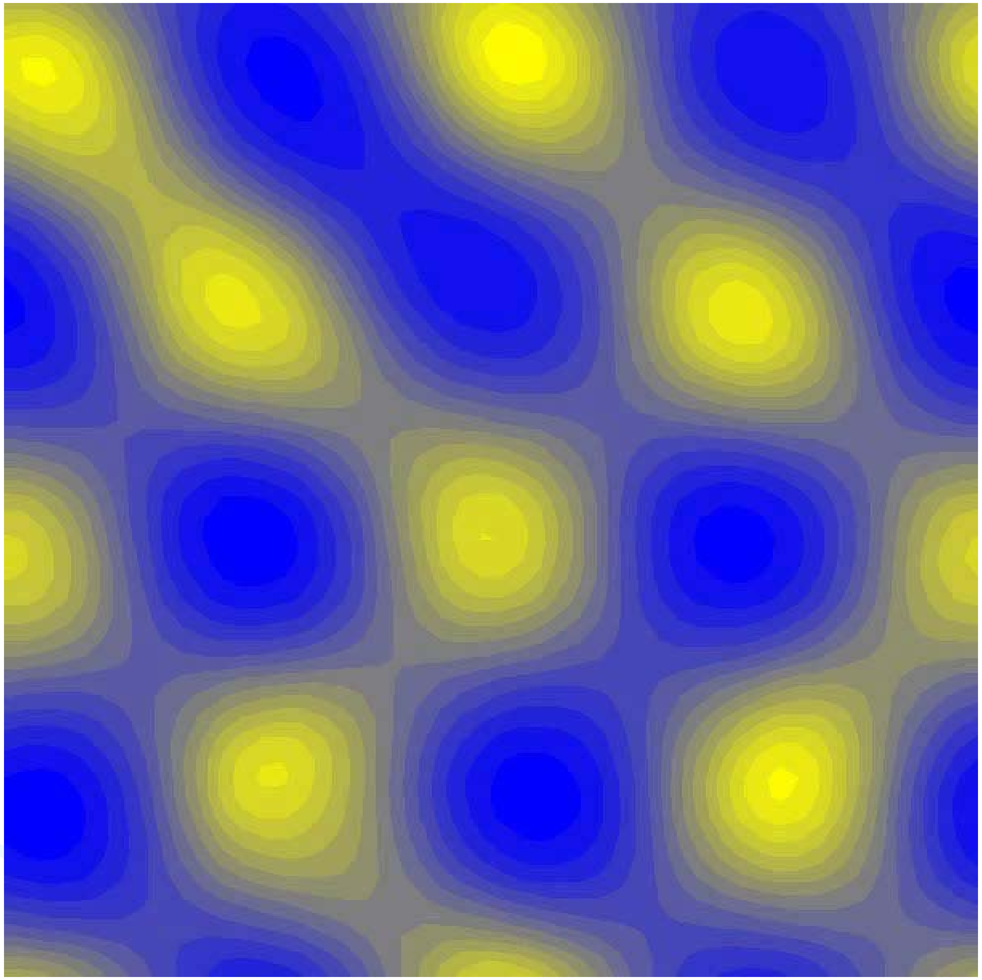


The *Ω*, *μ* plane. The domain corresponding to the wave instability is above the line. Other parameters of the model: ρ=0.23, c=1, d=1, $D_1 = 1$, $D_2 = 1$, $D_3 = 50$

Travelling waves (numerical simulations)

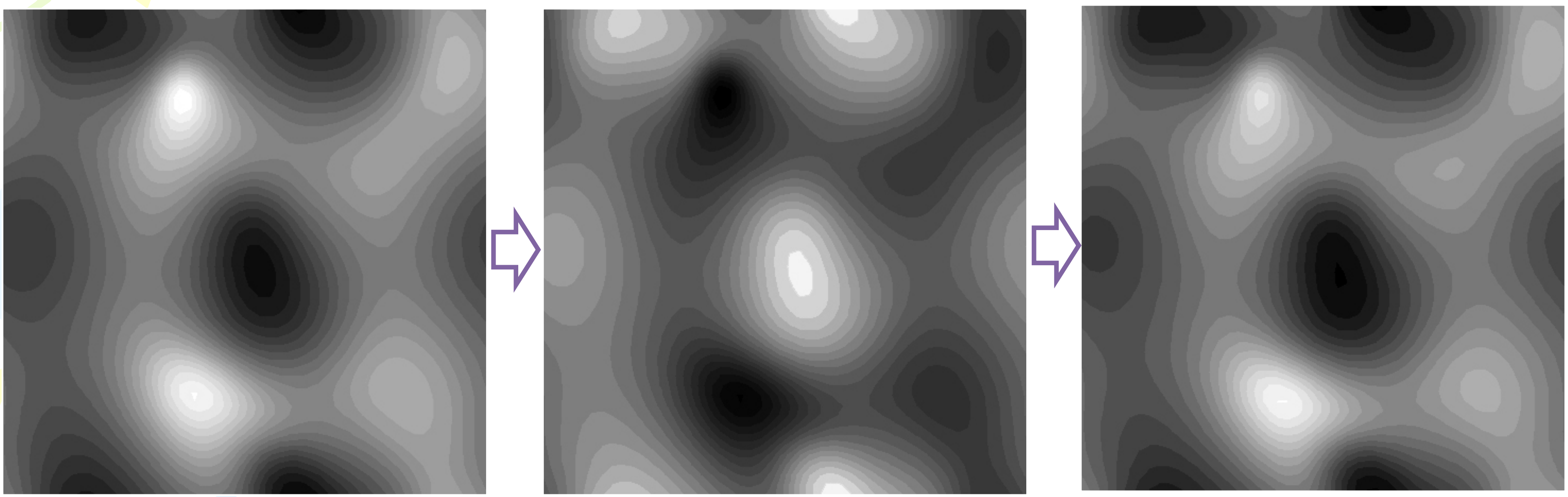


Parameters: $\rho=0.23$, $\mu=2$, $\Omega=3$, $c=1$, $d=1$, $D_1 = 1$, $D_2 =1$, $D_3 =50$.
Domain size: 150x150.

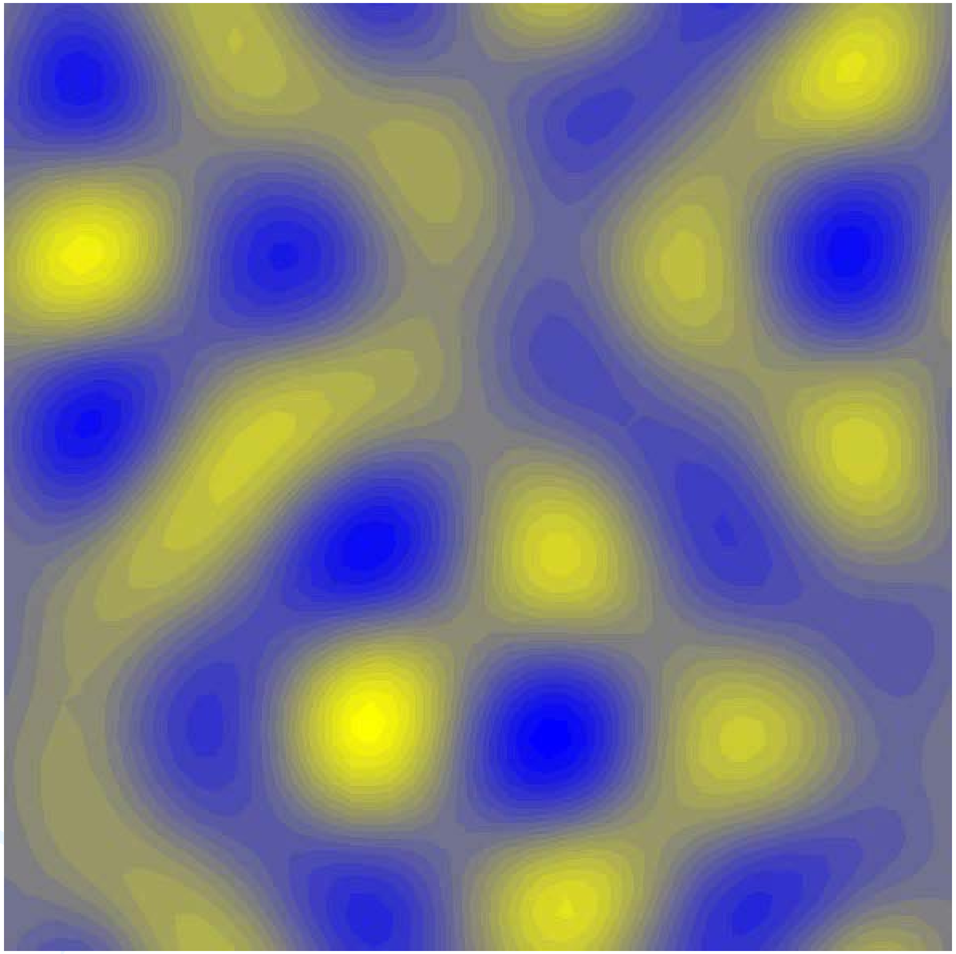


Travelling wave

Standing waves (numerical simulations)



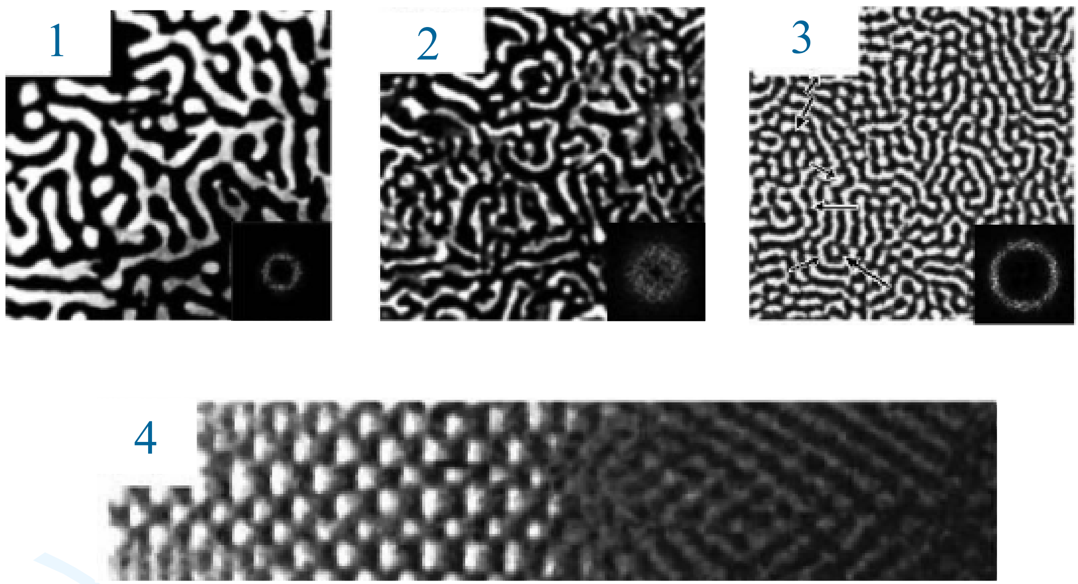
Parameters: ρ=0.23, μ=1.65, Ω=10, *c*=1, *d*=1, D_1 = 1, D_2 =1, D_3 =50.
Domain size: 150x150.



Standing wave

Mechanism of Switching From Standing to Traveling Waves

Accompanied by Halving of the Wavelength



[1]*Kaminaga A., Vanag V.K., Epstein I.R.* Wavelength Halving in a Transition between Standing Waves and Traveling Waves // *Phys. Rev. Lett*. v. 95, 2002.

Assumptions

- A standing wave is excited by a supercritical wave bifurcation for the wave with the wave vector k_{SW}

$$u_1(r,t) = \tilde{A}_1 e^{i(w_{SW}t + k_{SW}r)} + \tilde{A}_2 e^{i(w_{SW}t - k_{SW}r)}$$

$$\begin{cases} \dot{\tilde{A}}_1 = \tilde{A}_1 - (1 - ic_1)\tilde{A}_1\left|\tilde{A}_1\right|^2 - h_1(1 - ic_2)\tilde{A}_1\left|\tilde{A}_2\right|^2, \\ \dot{\tilde{A}}_2 = \tilde{A}_2 - (1 - ic_1)\tilde{A}_2\left|\tilde{A}_2\right|^2 - h_1(1 - ic_2)\tilde{A}_2\left|\tilde{A}_1\right|^2. \end{cases}$$

$h_1 \in (0,1)$ $\Rightarrow$ $|\tilde{A}_{1,2}| = \dfrac{1}{\sqrt{1+h_1}}$

- A wave with a twofold wave number $2k_{SW}$ related to a standing wave is stable, but can be excited in a rigid manner due to a subcritical bifurcation.

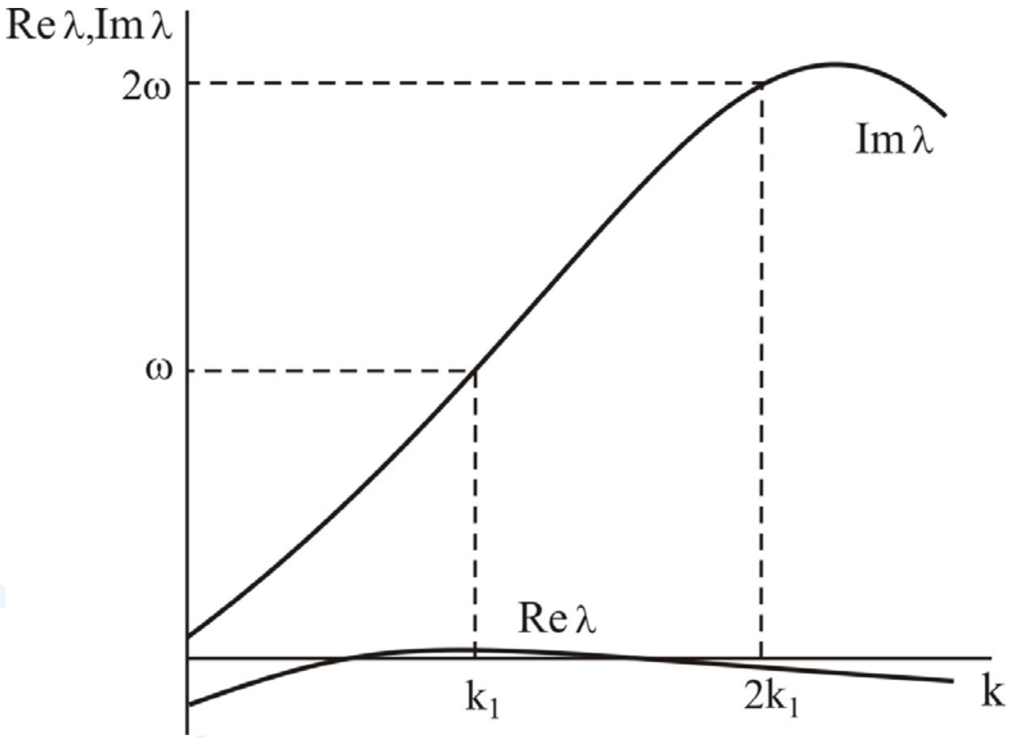
$$u_2(r,t) = \widetilde{B}_1 e^{i(w_2 t + 2k_{SW} r)} + \widetilde{B}_2 e^{i(w_2 t - 2k_{SW} r)}$$

In this case in the corresponding amplitude equations we should take into account besides cubic terms also terms of the fifth power.

$$\begin{cases} \dot{\widetilde{B}}_1 = -\alpha \widetilde{B}_1 + (1 - id_1)\widetilde{B}_1 \left|\widetilde{B}_1\right|^2 - (1 - id_3)\beta \cdot \widetilde{B}_1 \left|\widetilde{B}_1\right|^4 - h_2 (1 - id_2)\widetilde{B}_1 \left|\widetilde{B}_2\right|^2, \\ \dot{\widetilde{B}}_2 = -\alpha \widetilde{B}_2 + (1 - id_1)\widetilde{B}_2 \left|\widetilde{B}_2\right|^2 - (1 - id_3)\beta \cdot \widetilde{B}_2 \left|\widetilde{B}_2\right|^4 - h_2 (1 - id_2)\widetilde{B}_2 \left|\widetilde{B}_1\right|^2, \end{cases}$$

$$h_2 > 1 - 4\alpha\beta$$

- There is a resonance between the first and the second waves, namely, the wave with a twofold wave number has also a duplicated frequency.

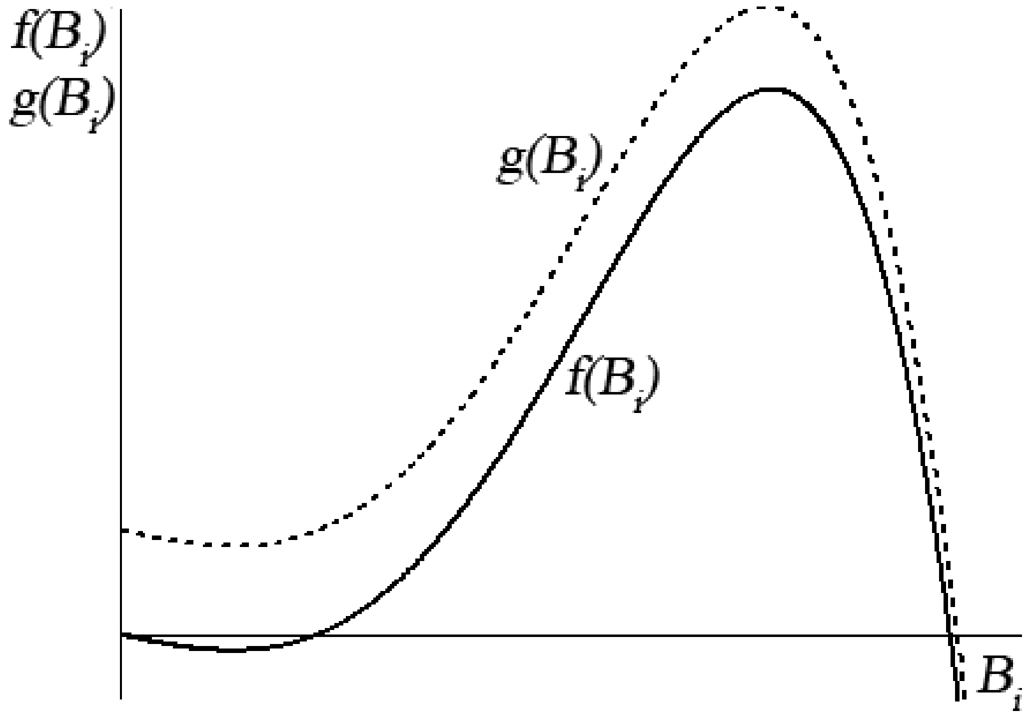


$$u(r,t)=\widetilde{A}_1 e^{i(w_{SW}t+k_{SW}r)}+\widetilde{A}_2 e^{i(w_{SW}t-k_{SW}r)}+\widetilde{B}_1 e^{i(2w_{SW}t+2k_{SW}r)}+\widetilde{B}_2 e^{i(2w_{SW}t-2k_{SW}r)}.$$

In terms of amplitude equations this resonance means that besides the conventional cubic terms describing interaction between the modes there is also a term, proportional to the square of the first mode amplitude in the equation for the second mode.

$$\begin{cases}\dfrac{dA_1}{dt}=A_1-A_1^3-h_1A_2^2A_1-\delta_1(B_1^2+B_2^2)A_1,\\ \dfrac{dA_2}{dt}=A_2-A_2^3-h_1A_1^2A_2-\delta_1(B_1^2+B_2^2)A_2,\\ \dfrac{dB_1}{dt}=-\alpha B_1+B_1^3-\beta B_1^5-h_2B_2^2B_1-\delta_2(A_1^2+A_2^2)B_1+\sigma\cos\psi_1(t)A_1^2,\\ \dfrac{dB_2}{dt}=-\alpha B_2+B_2^3-\beta B_2^5-h_2B_1^2B_2-\delta_2(A_1^2+A_2^2)B_2+\sigma\cos\psi_2(t)A_2^2,\end{cases}$$

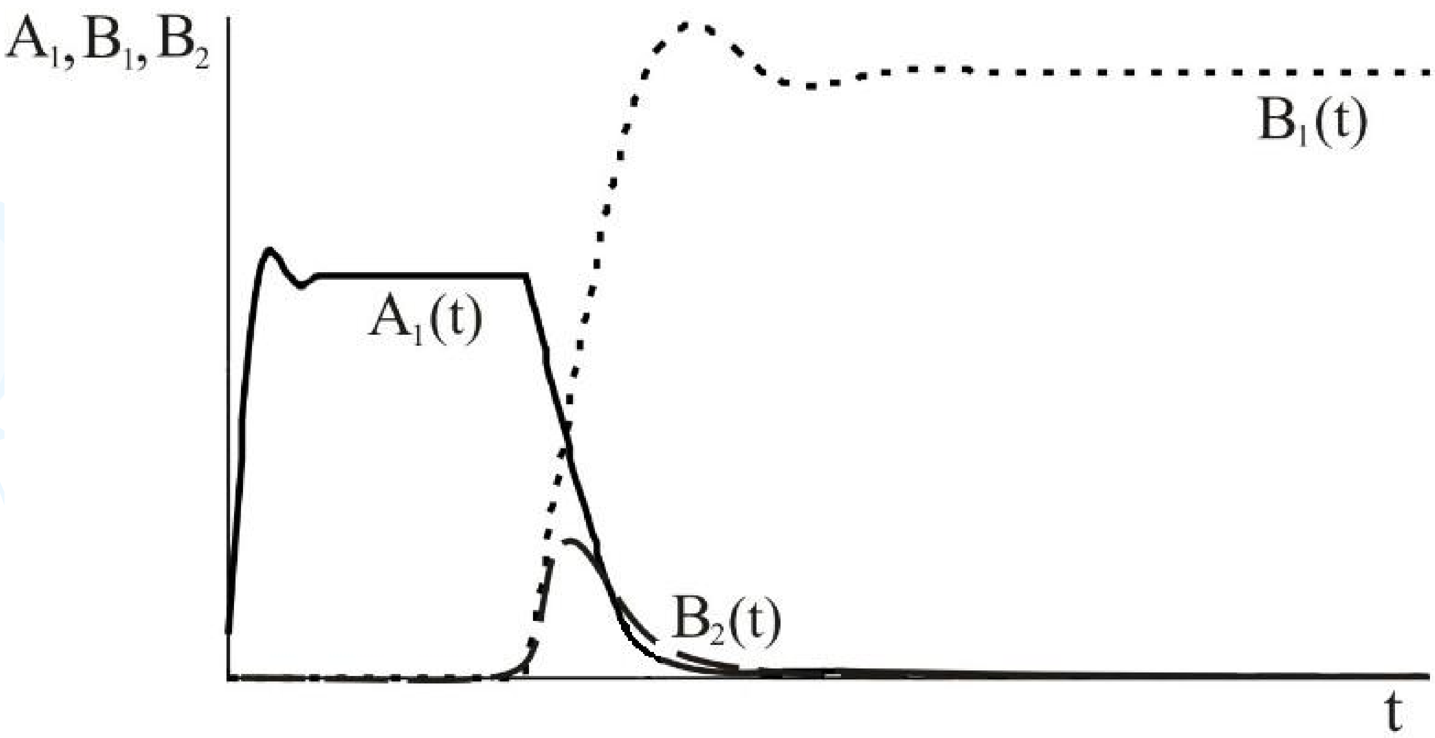
where $A_i=\left|\widetilde{A}_i\right|$ and $B_i=\left|\widetilde{B}_i\right|$



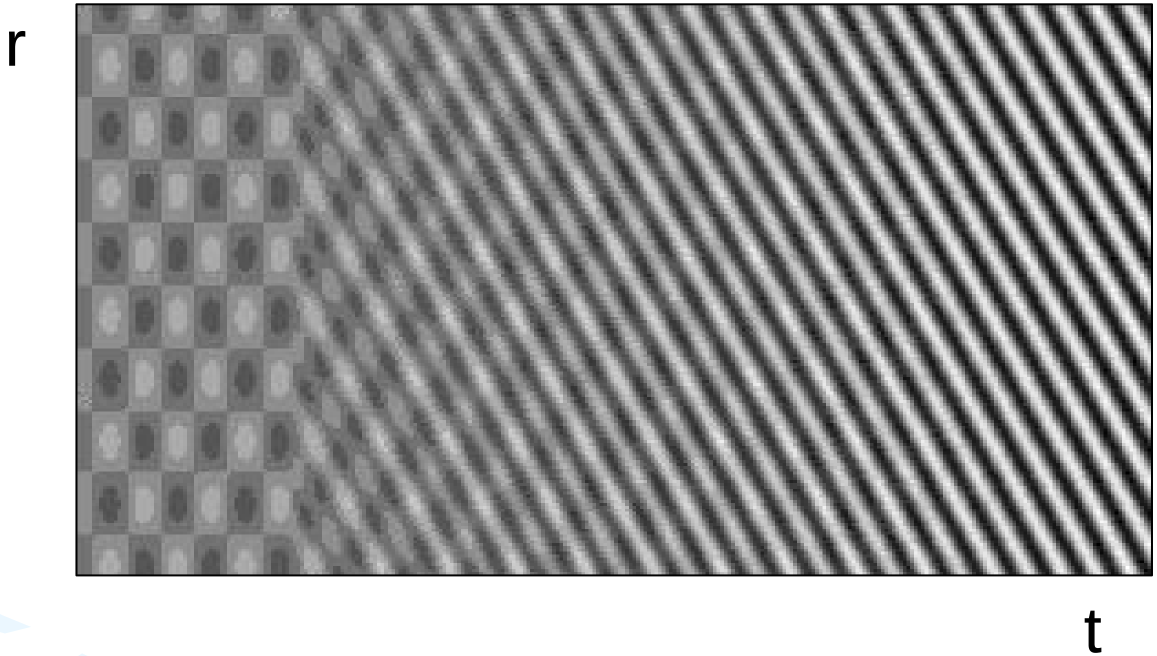
$$f(B_i) = -\alpha B_i + B_i^3 - \beta B_i^5$$

$$g(B_i) = -\left(\alpha + 2\frac{\delta_2}{1+h_1}\right)B_i + B_i^3 - \beta B_i^5 + \frac{\sigma}{1+h_1}$$

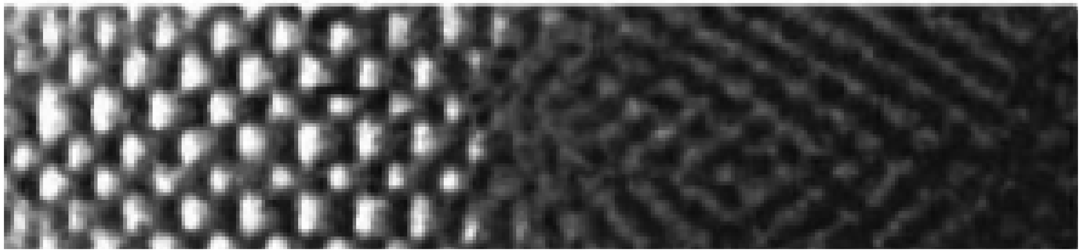
Numerical experiments



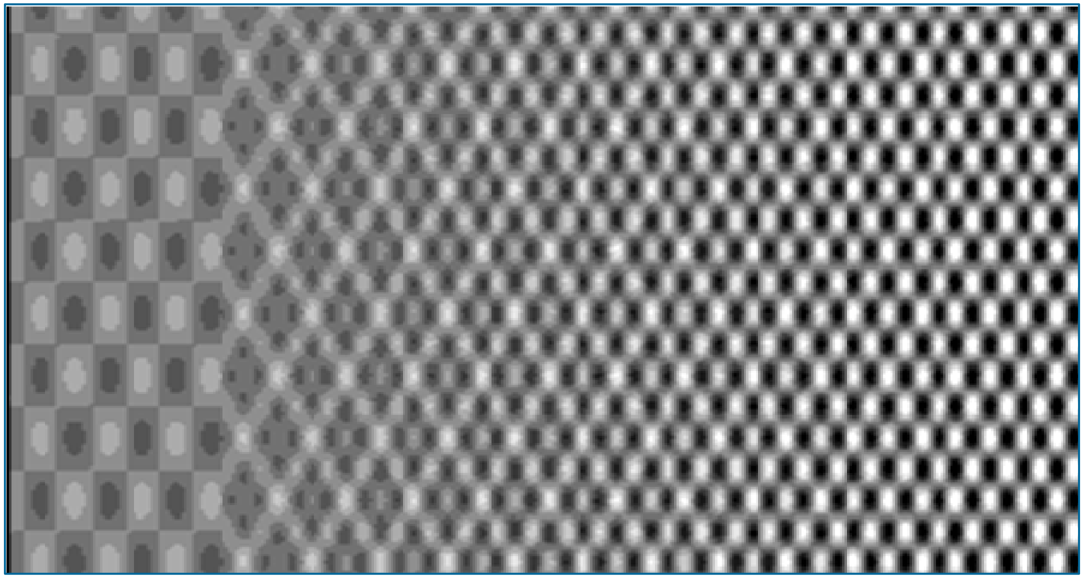
Space-time plot *u(r,t)* of the transition from standing to travelling wave for $h_2 > 1 - 4\alpha\beta$



experiment



Space-time plot *u(r,t)* of the transition from standing to **standing** wave with half wavelength for $h_2 < 1 - 4\alpha\beta$



Conclusion

- We have shown that though in multidimensional space a lot of modes contribute to pattern formation, in fact the variety of possible patterns is very limited: either there is a standing wave with rather complicated spatial structure (all of the modes survive) for low intermodal competition, or there is a quasi one-dimensional travelling wave for strong competition. In fact it means that having just a snapshot of the pattern we can immediately say whether we deal with a standing or travelling wave.

- We suggested a possible mechanism for a rather nontrivial phenomenon observed in experiment: the transition from standing waves to travelling waves with the half-wavelength, based on the hypothesis of a kind of resonance between the unstable mode, responsible for the standing wave, and the rigidly exited mode with a twofold wave number. Though, from the point of view of the theory of dynamical systems, this situation is noncoarse, it is possible that due to the drift of parameters in a real experimental system such resonance may occur, resulting in the observed transition.

Thank you for your attention!