Comments about Higher-Spin and Duality. Part II: Mixed-Symmetry Fields and Unfolding

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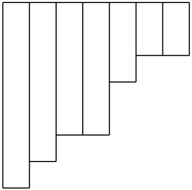
N. Boulanger, P. P. Cook, D. P. [arXiv:1205.2277[hep-th]],
N. Boulanger, D. P. [arXiv:1205.????[hep-th]].

Overview and Key Ingredients. Mixed-Symmetry Fields.

- Massless mixed-symmetry field in Minkowski space:

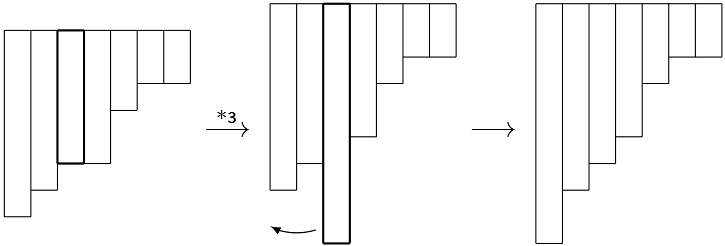
$$(\Box + \dots)$$  $$= 0$$

- Frame-like formulation:
  Generalisation of Cartan formulation of gravity in terms of frame field e and spin-connection ω
- Unfolding:

$$\text{Unfolding} = \text{Frame-like} + \text{Integrability}.$$

Overview and Key Ingredients. Dualisation

- Duality:
  On-shell representation is given by a tensor of the Wigner little group $SO(d-2)$. Hodge duality maps $h_i \to d-2-h_i$.



- Parent action approach:

Parent: $S(\text{initial fields}, \text{dual fields})$

$\swarrow \qquad \searrow$

Children: $S(\text{initial fields})$ $\quad$ $S(\text{dual fields})$

Motivation

- Bargmann-Wigner-Fierz-Pauli programm: fields are not traceless fields on-shell.
- Transformation, relating child theories through the parent action can be implemented in the path integral, thus establishing equivalence at the quantum level
  [Fradkin, Tseytlin '85]
- Different fields dual to gravity appear in the context of Kac-Moody algebra E_{11}
  [West '01], [Ricconi, West '06]
- The duality between an exotic $6d$ $(4,0)$ superconformal theory and the strong coupling limit of maximally supersymmetric $\mathcal{N}=8$ supergravity in $5d$ was conjectured. The dual and the double dual gravitons appear via dimensional reduction of $6d$ theory.
  [Hull '00,'01]

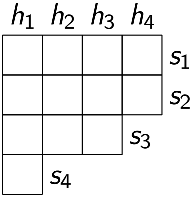
Plan

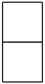
- Frame-like massless mixed symmetry fields in Minkowski space
- Dualisation generalities
- Frame-like parent action dualisation of
  - gravity $\to$ dual gravity
  - dual gravity $\to$ double-dual gravity
  - General Massless Mixed Symmetry field

Massless Mixed-Symmetry Fields in Minkowski space

Unitary irreducible representation of $ISO(d-1,1)$ is induced from a tensorial irreducible representation of Wigner's little group $SO(d-2)$. Irreducibility under $SO(d-2)$ requires *symmetry* and *trace* constraints.

Symmetry constraints: Young diagrams

$$\mathbf{Y}[h_1, h_2, h_3, h_4] = \mathbf{Y}(s_1, s_2, s_3, s_4) =$$


For example  and  denote sym. and antisym. rank 2 tensors.

Trace constraints: Tracelessness. $h_1 + h_2 > d$ are not allowed for $SO(d)$.

Frame-Like Gravity

$g_{\mu\nu} \longrightarrow (e, \omega)$,
${e_\mathbf{1}}^a$ — frame field, provides tangent fiber space basis with flat metric η_{ab};
1-form valued in $\mathbf{Y}[1]$-shaped fiber tensors.
${\omega_\mathbf{1}}^{ab}$ — spin-connection, defines parallel transport in fiber space;
1-form valued in $\mathbf{Y}[2]$-shaped fiber tensors.

$$T_\mathbf{2}^a = de_\mathbf{1}^a + \omega_\mathbf{1}^a{}_b e_\mathbf{1}^b, \quad R_\mathbf{2}^{ab} = d\omega_\mathbf{1}^{ab} + \omega_\mathbf{1}^a{}_c \omega_\mathbf{1}^{cb}.$$

$$S = \int R_\mathbf{2}^{a_1 a_2} e_\mathbf{1}^{a_3} \ldots e_\mathbf{1}^{a_d} \varepsilon_{a_1 \ldots a_d}.$$

$$\frac{\delta S}{\delta \omega} \propto T = 0, \quad \frac{\delta S}{\delta e} \propto Tr[R] = 0.$$

Unfolded Linear Gravity

Linearization around the flat background: $e \to h + e$, $\omega \to \varpi + \omega$.

$$T^a_{\mathbf{2}bg} = dh^a_{\mathbf{1}} + \varpi^a_{\mathbf{1}\,b} h^b_{\mathbf{1}} = 0, \quad R^{ab}_{\mathbf{2}bg} = d\varpi^{ab}_{\mathbf{1}} + \varpi^a_{\mathbf{1}\,c} \varpi^{cb}_{\mathbf{1}} = 0.$$

Linearized action is

$$S = \int (de^{a_1}_{\mathbf{1}} + \frac{1}{2}\omega^{a_1}_{\mathbf{1}\ b} h^b_{\mathbf{1}}) \omega^{a_2 a_3}_{\mathbf{1}} h^{a_4}_{\mathbf{1}} \dots h^{a_d}_{\mathbf{1}} \varepsilon_{a_1 \dots a_d} \equiv \langle de + \frac{1}{2}\omega h | \omega \rangle.$$

Equations of motion are

$$T^a = de^a_{\mathbf{1}} + \omega^a_{\mathbf{1}\,b} h^b_{\mathbf{1}} = 0, \quad Tr[d\omega^{ab}_{\mathbf{1}}] = 0.$$

the 2-nd equation can be rewritten

$0 = d\omega^{ab}_{\mathbf{1}} + h_{\mathbf{1}c} h_{\mathbf{1}d} C^{abcd}_{\mathbf{0}}$, where $C^{abcd}_{\mathbf{0}}$ is $\mathbf{Y}[2,2]$ – shaped Weyl tensor.

Unfolded Linear Gravity

One can continue by writing differential equations for Weyl tensor

$$0 = dC_{\mathbf{0}}^{abcd} + \Pi(h_{\mathbf{1}e} C_{\mathbf{0}}^{abcde}), \quad \text{where } C_{\mathbf{0}}^{abcde} \text{ is } \mathbf{Y}[2,2,1] - \text{shaped tensor},$$

$$0 = dC_{\mathbf{0}}^{abcde} + \Pi(h_{\mathbf{1}e} C_{\mathbf{0}}^{abcdef}), \quad \text{where } C_{\mathbf{0}}^{abcdef} \text{ is } \mathbf{Y}[2,2,1,1] - \text{shaped tensor}.$$

$$\dots$$

Summarizing the result, unfolded equations for linear gravity are

$$0 = dW_{\mathbf{p^i}}^{\mathbf{Y}^i} + \sigma_-(h_{\mathbf{1}}) W_{\mathbf{p^{i+1}}}^{\mathbf{Y}^{i+1}} \equiv dW^i + \sigma_- W^{i+1},$$

where i is the grade enumerating fields, $\sigma_-(h_{\mathbf{1}})$ is defined uniquely by p^{i+1}, $\mathbf{Y}^{i+1}$, p^i and $\mathbf{Y}^i$, $\sigma_-^2 = 0 \Leftrightarrow$ integrability. Gauge symmetries

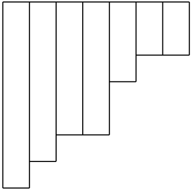
$$W_{\mathbf{p^i}}^{\mathbf{Y}^i} \quad \to \quad \varepsilon_{\mathbf{p^i-1}}^{\mathbf{Y}^i} \quad \to \quad \bar{\varepsilon}_{\mathbf{p^i-2}}^{\mathbf{Y}^i} \quad \to \quad \dots$$

$$S = \langle dW^1 + \frac{1}{2}\sigma_- W^2 | W^2 \rangle.$$

Unfolded Massless Mixed-Symmetry Fields

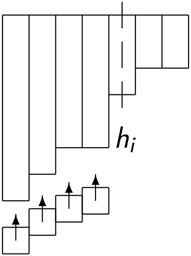
Massless spin-$\mathbf{Y}$ field on-shell describes $SO(d-2)$ traceless $\mathbf{Y}$-shaped tensor

$$\mathbf{Y}[h_1, h_2, \ldots, 0, 0 \ldots] =$$



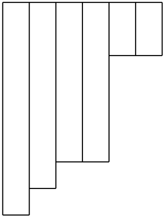
Then

$$\mathbf{Y}^i =$$



h_i

$$=$$



$$, \qquad p^i = h_i.$$

[Skvortsov '08]

Dualisation Generalities and "Pure Traces"

Given spin-$\mathbf{Y}[h_1, h_2, \ldots]$ field on-shell we can perform:

- The first column dualisation: $\mathbf{Y}[h_1, h_2, \ldots] \to \mathbf{Y}'[d-2-h_1, h_2, \ldots]$,

$$h_1 \geq h_2 \quad \Leftrightarrow \quad (d-2-h_1) + h_2 \leq d-2$$

$$h_1 + h_2 \leq d-2 \quad \Leftrightarrow \quad d-2-h_1 \geq h_2.$$

  The dual diagram is allowed.

- The i-th column dualisation $(i \neq 1)$,
  $\mathbf{Y}[h_1, \ldots, h_i, \ldots] \to \mathbf{Y}''[d-2-h_i, h_1, \ldots]$,

$$h_1 \geq h_i \quad \Leftrightarrow \quad h_1 + (d-2-h_i) \geq d-2.$$

$$h_1 + h_i \leq d-2, \quad \Leftrightarrow \quad h_1 \leq d-2-h_i,$$

  The dual diagram is not allowed. This implies that traceless tensor maps to *pure trace* tensor of the form $\eta \ldots \eta C$ with traceless C.

Dualisation Generalities and "Pure Traces"

Examples of on-shell dualisation.

- Gravity on-shell is given by the traceless symmetric rank 2 tensor h_{ij}. Dual graviton is defined on-shell

$$C^{m[d-3],}{}_{p} = \varepsilon^{m[d-3]n} h_{np}.$$

C is traceless $\mathbf{Y}[d-3,1]$-shaped tensor.

- The second dualisation of gravity

$$Y^{m[d-3],n[d-3]} = \varepsilon^{m[d-3]p} C^{n[d-3],}{}_{p} = \varepsilon^{m[d-3]p} \varepsilon^{n[d-3]r} h_{pr},$$

$$Y^{m[d-3],}{}_{n[d-3]} = \sigma(d-2)! \delta^{m[d-3]r}_{n[d-3]p} h^{p}{}_{r} =$$

$$\sigma(d-2)! \left(\frac{1}{d-2} \delta^{m[d-3]}_{n[d-3]} \delta^{r}_{p} h^{p}{}_{r} - \frac{d-3}{d-2} \delta^{[m[d-4]}_{[n[d-4]} h^{m]}{}_{n]} \right).$$

Y is $(d-4)$-fold pure-trace.

Dualisation of Linear Gravity. Parent Action

The parent action for the first dualisation is

$$S=\int\left[(de_{\mathbf{1}}^{a_1}+\frac{1}{2}h_{\mathbf{1}}^{b}\omega_{\mathbf{1}}^{a_1}{}_{b}+t_{\mathbf{2}}^{a_1})\omega_{\mathbf{1}}^{a_2a_3}h_{\mathbf{1}}^{a_4}\dots h_{\mathbf{1}}^{a_d}\varepsilon_{a_1\dots a_d}+t_{\mathbf{2}a}d\tilde{e}_{\mathbf{d-3}}^{a}\right],$$

$t_{\mathbf{2}}^{a}$ is torsion-like field, $\tilde{e}_{\mathbf{d-3}}^{a}$ is frame-like dual field.
Gauge symmetries are

$$\delta e_{\mathbf{1}}^{a}=d\xi_{\mathbf{0}}^{a}+h_b\lambda_{\mathbf{0}}^{ab}-\psi_{\mathbf{1}}^{a},\quad \delta\xi_{\mathbf{0}}^{a}=\bar{\psi}_{\mathbf{0}}^{a},$$

$$\delta\omega_{\mathbf{1}}^{a[2]}=d\lambda_{\mathbf{0}}^{a[2]},$$

$$\delta t_{\mathbf{2}}^{a}=d\psi_{\mathbf{1}}^{a},\quad \delta\psi_{\mathbf{1}}^{a}=d\bar{\psi}_{\mathbf{0}}^{a},$$

$$\delta\tilde{e}_{\mathbf{d-3}}^{a}=d\tilde{\xi}_{\mathbf{d-4}}^{a}-h_{l[d-3]}(*\lambda_{\mathbf{0}})^{al[d-3]},\quad \delta\tilde{\xi}_{\mathbf{d-4}}^{a}=d\tilde{\bar{\xi}}_{\mathbf{d-5}}^{a},\quad \dots$$

Child Actions

$$S = \langle de + \frac{1}{2}\sigma_{-}\omega + t|\omega\rangle + t \cdot d\tilde{e}.$$

- Equivalence to gravity

$$\frac{\delta S}{\delta \tilde{e}} = 0 \quad \Rightarrow \quad dt^a_{\mathbf{2}} = 0 \quad \Rightarrow \quad t^a_{\mathbf{2}} = d\beta^a_{\mathbf{1}}.$$

$t^a_{\mathbf{2}}$ can be gauged away by $\psi^a_{\mathbf{1}}$-symmetry giving the linearised gravity child action.

- Dual gravity child action. We gauge away e by ψ-symmetry, $e = 0$.

$$\frac{\delta S}{\delta \omega} = 0 \quad \Rightarrow \quad de^a_{\mathbf{1}} + h_b\omega^{ab}_{\mathbf{1}} + t^a_{\mathbf{2}} = 0, \quad \Rightarrow \quad t^a_{\mathbf{2}} = -h_b\omega^{ab}_{\mathbf{1}}.$$

$S = \langle d\tilde{e} + \frac{1}{2}\sigma_{-}\omega|\omega\rangle$ which is Skvortsov's action for $\mathbf{Y}[d-3,1]$.

Unfolded equations for the dual theory has the same Weyl module

Dual Gravity $\longrightarrow$ Gravity

More explicitly, the dual gravity action is

$$S = \int (de^a_{\mathbf{d-3}} + \frac{1}{2} h_{b[d-3]} \omega_{\mathbf{1}}^{ab[d-3]}) \omega_{\mathbf{1}}^{c[d-2]} h^l \varepsilon_{ac[d-2]l}.$$

We can act in the similiar way: we add an auxiliary field t such that associated gauge symmetry acts algebraically on e and allows to gauge it away; we introduce the dual field via the term $t \cdot \tilde{e}$.

$$S = \int \Big[(de^a_{\mathbf{d-3}} + \frac{1}{2} h_{b[d-3]} \omega_{\mathbf{1}}^{ab[d-3]} + t^a_{\mathbf{d-2}}) \omega_{\mathbf{1}}^{c[d-2]} h^l \varepsilon_{ac[d-2]l} + t^a_{\mathbf{d-2}} d\tilde{e}^a_{\mathbf{1}} \Big].$$

It leads back to linearized gravity.

Dual Gravity ⟶ Double-Dual Gravity

$$S = \int \Big[(de^a_{\mathbf{d-3}} + \frac{1}{2} h_{b[d-3]}\omega_{\mathbf{1}}^{ab[d-3]} + h_{b[d-4]} t_{\mathbf{2}}^{ab[d-4]})\omega_{\mathbf{1}}^{c[d-2]} h^l \varepsilon_{ac[d-2]l} +$$

$$(-1)^{d-1} t_{\mathbf{2}a[d-3]} d\tilde{e}^{a[d-3]}_{\mathbf{d-3}} + (-1)^{d-1}\frac{\alpha}{2} t_{\mathbf{2}a[d-3]} h^{[a[d-5]} h_c (*t)_{\mathbf{2}}^{a[2]]c} \Big],$$

α is arbitrary coefficient.
Equivalence to the dual gravity action:

$$\frac{\delta S}{\delta \tilde{e}} = 0 \quad \Rightarrow \quad dt_{\mathbf{2}}^{a[d-3]} = 0 \quad \Rightarrow \quad t_{\mathbf{2}}^{a[d-3]} = d\beta_{\mathbf{1}}^{a[d-3]}.$$

Then $t_{\mathbf{2}}^{a[d-3]}$ can be gauged away. We reproduce the dual gravity action.

Double-Dual Gravity

Unfolded equations:

$$T^a_{\mathbf{d-2}} := de^a_{\mathbf{d-3}} + h_{b[d-3]}\omega_{\mathbf{1}}^{ab[d-3]} + h_{b[d-4]}t_{\mathbf{2}}^{ab[d-4]} = 0,$$

$$\tilde{T}^{a[d-3]}_{\mathbf{d-2}} := d\tilde{e}^{a[d-3]}_{\mathbf{d-3}} + h^{[a[d-4]}h_b(*\omega_{\mathbf{1}})^{a]b} + \alpha h^{[a[d-5]}h_b(*t)^{a[2]]b}_{\mathbf{2}} = 0,$$

$$R^{a[d-2]}_{\mathbf{2}} := d\omega^{a[d-2]}_{\mathbf{1}} + h_b h_b C^{a[d-2],b[2]}_{\mathbf{0}} = 0,$$

$$\tau^{a[d-3]}_{\mathbf{3}} := dt^{a[d-3]}_{\mathbf{2}} = 0 \quad + \quad \text{eqs. for the Weyl module}$$

Gauge symmetries

$$\delta e^a_{\mathbf{d-3}} = d\xi^a_{\mathbf{d-4}} + (-1)^{d-2}h_{b[d-3]}\lambda^{ab[d-3]}_{\mathbf{0}} + (-1)^{d-3}h_{b[d-4]}\psi^{ab[d-4]}_{\mathbf{1}},$$

$$\delta\omega^{a[d-2]}_{\mathbf{1}} = d\lambda^{a[d-2]}_{\mathbf{0}}, \quad \delta t^{a[d-3]}_{\mathbf{2}} = d\psi^{a[d-3]}_{\mathbf{1}},$$

$$\delta\tilde{e}^{a[d-3]}_{\mathbf{d-3}} = d\tilde{\xi}^{a[d-3]}_{\mathbf{d-4}} + (-1)^{d-3}h^{[a[d-4]}h_b(*\lambda_{\mathbf{0}})^{a]b} +$$

$$(-1)^{d-4}\alpha h^{[a[d-5]}h_b(*\psi)^{a[2]]b}_{\mathbf{1}}.$$

Double-Dual Gravity

As planned, $e^a_{\mathbf{d-3}}$ can be gauged away by the algebraic $\psi_{\mathbf{1}}^{ab[d-4]}$-symmetry.

- However, in contrast to the first dualisation, ξ-symmetry cannot be "gauged away" by the second order gauge symmtry $\bar{\psi}$

$$\delta\xi^a_{\mathbf{d-4}} = d\bar{\xi}^a_{\mathbf{d-5}} + h_{b[d-4]}\bar{\psi}_{\mathbf{0}}^{ab[d-4]}.$$

  It acts on the double-dual graviton, producing divergence-like gauge symmetry.

- The second important difference is that auxiliary field t cannot be expressed in terms of the connection-like field ω and thus it cannot be excluded from the action.
- Note arbitrary parameter α.

The parent action given is the most economical form of the frame-like action for the double-dual gravity. Nonetheless, it is possible to write metric-like action that contains smaller set of fields.

`[Boulanger, Cook, D.P. '12]`

Dualisation of Arbitrary Mixed symmetry field

The parent action for i-th column dualisation is

$$S = \langle de + \frac{1}{2}\sigma_{-}\omega + \Sigma_{-} t | \omega\rangle + t \cdot d\tilde{e} + \alpha t^2 .$$

For spin-$\mathbf{Y}[h_1, h_2, \ldots, h_i, \ldots]$ field

- e is h_1-form valued in $\mathbf{Y}[h_2, \ldots, h_i, \ldots]$,
- ω is h_2-form valued in $\mathbf{Y}[h_1 + 1, \ldots, h_i, \ldots]$,
- t is $h_i + 1$-form valued in $\mathbf{Y}[h_1, \ldots, h_{i-1}, h_{i+1} \ldots]$,
- $\tilde{e}$ is $d - h_i - 2$-form valued in $\mathbf{Y}[h_1, \ldots, h_{i-1}, h_{i+1} \ldots]$.

Σ_{-} is defined uniquely by its properties. αt^2 contains all possible contractions bilinear in t. For the dualisation on the first column the αt^2 terms are absent (cannot be constructed).

Conclusion

- Frame-like parent action principle for arbitrary massless mixed-symmtry fields has been built
- Dual theories describe proper degrees of freedom which is ensured by the unfolding machinery as well as by the parent action approach
- Some dualisations result in theories for fields not traceless on-shell. Actions for these theories has not been known before. We observed and studied peculiarities of these pure trace theories such as diveregence-like gauge transformations and free coefficients.
- Simple frame-like form of the action is suggestive for aplications and generalisations.

Thank you!

Unfolded equations

Generalized curvatures R^α for fields W^α

$$R^\alpha(x) \stackrel{def}{=} dW^\alpha(x) + G^\alpha(W(x)),$$

$$G^\alpha(W^\beta) \stackrel{def}{=} \sum_{n=1}^{\infty} f^\alpha_{\beta_1\ldots\beta_n} W^{\beta_1}\ldots W^{\beta_n}.$$

Compatibility condition

$$G^\beta(W)\frac{\delta^L G^\alpha(W)}{\delta W^\beta} \equiv 0 \quad \Rightarrow \quad dR^\alpha \equiv R^\beta \frac{\delta G^\alpha}{\delta W^\beta}.$$

Does not depend on the base space

Equations:

$$R^\alpha(x) = 0.$$

Gauge invariance

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta \frac{\delta^L G^\alpha(W)}{\delta W^\beta}.$$

Examples

Zero curvature equation in YM theory

Field: 1-form $\hat{\Omega}_0 = \Omega_0^a \hat{T}_a \in g$, g — Lie algebra, $\hat{T}_a$ — generators.

$$\hat{R} \stackrel{def}{=} d\hat{\Omega}_0 + \hat{\Omega}_0\hat{\Omega}_0 = 0.$$

Let g be Poincare algebra $(\hat{P}_a, \hat{M}_{ab})$

$$\hat{\Omega}_0 = e_0{}^a \hat{P}_a + \omega_0{}^{ab} \hat{M}_{ab}, \quad \hat{R} = R_L{}^{ab} \hat{M}_{ab} + T^a \hat{P}_a,$$

$$R_L{}^{ab} = d\omega_0^{ab} + \omega_0{}^{ac} \omega_{0c}{}^b = 0,$$

$$T^a = de_0^a + \omega_0{}^{ab} e_{0b} = 0.$$

Describes background geometry of Minkowski space.

Examples

Covariant constancy equation

Fields: set of p-forms C^i, $W = \Omega_0 + C + \dots$.

$$R^i \stackrel{def}{=} dC^i + \Omega_0^a (T_a)^i{}_j C^j = 0.$$

Representation

$$\hat{A} = A^a \hat{T}_a \quad \rightarrow \quad A^i{}_j = A^a (T_a)^i{}_j.$$

$C \in$ representation space,

$$R^i = 0 \quad \Leftrightarrow \quad D_{\Omega_0} C^i = 0.$$

Examples

Massless scalar field

Cartesian coordinates

$$e_{0m}{}^{a} = \delta^{a}_{m}, \quad \omega_{0m}{}^{ab} = 0, \quad D^{L} = d.$$

Fields: 0-forms $C^{a(k)}$, $C_b{}^{ba(k-2)} = 0$.
Curvatures:

$$R^{a(k)} \stackrel{def}{=} dC^{a(k)} + e_0{}^{b} C_b{}^{a(k)}.$$

The first and the second equations

$$\partial_a C(x) + C_a(x) = 0,$$

$$\partial_b C_a(x) + C_{ab}(x) = 0$$

entail

$$C_b{}^{b}(x) = 0 \quad \Rightarrow \quad \partial^a \partial_a C(x) = 0.$$

σ_-–cohomology technics

Unfolded equations for p-forms C

$$R \stackrel{def}{=} (d + \sum \sigma)C = 0,$$

σ — algebraic operators.

$$\delta C = (d + \sum \sigma)\varepsilon,$$

$$I = (d + \sum \sigma)R \equiv 0$$

— gauge symmetries and Bianchi identities.
How to analyse them? How to identify dynamical fields and dynamical equations?

σ_-–cohomology technics

The choice of dynamical fields is not unique!

Example

$$\frac{\partial}{\partial x} B(x) + A(x) = 0, \quad \frac{\partial}{\partial x} A(x) + B(x) = 0.$$

Introduce $\mathbb{Z}$-grade $\mathcal{G}$: diagonalizable on the space of fields, bounded below.

$$R \overset{def}{=} (d + \sigma_-)C = 0,$$

$$\delta C = (d + \sigma_-)\varepsilon,$$

$$I = (d + \sigma_-)R \equiv 0,$$

σ_- — the only algebraic operator, lowers grade.
Compatibility conditions $\Rightarrow \quad (\sigma_-)^2 = 0$.

σ_-–cohomology technics.

Dynamical fields

Expressing fields in terms of fields of lower grade by means of

$$(d + \sigma_-)C = 0, \qquad \{dC^{k-1} + \sigma_-^1 C^k = 0\}$$

we get $C \notin \mathrm{Ker}(\sigma_-) \Rightarrow C$ – auxiliary.
Gauge transformation

$$\delta C = (d + \sigma_-)\varepsilon$$

allows to fix $C = 0$ if $C \in \mathrm{Im}(\sigma_-)$.
Result: dynamical fields C_d

$$C_d \in \frac{\mathrm{Ker}(\sigma_-)}{\mathrm{Im}(\sigma_-)} = H(\sigma_-).$$

σ_-–cohomology technics. The result

Dynamical fields C_d

$$C_d \in \frac{\mathrm{Ker}_p(\sigma_-)}{\mathrm{Im}_p(\sigma_-)} = H_p(\sigma_-),$$

Dynamical equations $R_d = 0$

$$R_d \in \frac{\mathrm{Ker}_{p+1}(\sigma_-)}{\mathrm{Im}_{p+1}(\sigma_-)} = H_{p+1}(\sigma_-),$$

Differential gauge symmetries ε_d (cannot be fixed by algebraic gauge)

$$\varepsilon_d \in \frac{\mathrm{Ker}_{p-1}(\sigma_-)}{\mathrm{Im}_{p-1}(\sigma_-)} = H_{p-1}(\sigma_-),$$

where p — rank of C as a differential form.
[O.V. Shaynkman and M.A. Vasiliev '00]

$H(\sigma_-)$-analysis for massless scalar field

Curvatures:

$$R^{a(k)} \stackrel{def}{=} dC^{a(k)} + e_0{}^b C_b{}^{a(k)}.$$

Grade $\mathcal{G}$ counts number of tensor indices, σ_- — contraction with frame e_0^b, $[\mathcal{G}, \sigma_-] = -\sigma_-$.

Dymanical field

only $C \in \mathrm{Ker}_0(\sigma_-)$, $\mathrm{Im}_0(\sigma_-) = 0 \Rightarrow C \in \mathrm{H}_0(\sigma_-)$.

Dynamical equation

has the form $R_d \sim e_0^a t$, where t is 0-form.
Indeed, $e_0{}^b e_{0b} t = 0$, $e_0^a t \notin \mathrm{Im}_1(\sigma_-) \Rightarrow e_0^a t \in \mathrm{H}_1(\sigma_-)$. Then $t \sim R_m{}^m = \partial_m C^m = 0$ yields dynamical equation.

[O.V. Shaynkman and M.A. Vasiliev '00]