Stability of de Sitter Solutions in Modified Gravity

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Modern cosmological observations indicate that the current expansion of the Universe is accelerating.

The simplest model able to reproduce this late-time cosmic acceleration is general relativity with a cosmological constant.

There are a lot of different models of modified gravity (review [1]).

[1] M. Kilbinger *et al.*, *Dark energy constraints and correlations with systematics from CFHTLS weak lensing, SNLS supernovae Ia and WMAP5*, *Astron. Astrophys.* **497** (2009) 677–688 [arXiv:0810.5129]

Nonlocal gravitational model

We consider model that include a function of the $\Box^{-1}$ operator.

Action for nonlocal gravity

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\Box^{-1}R)\right)-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\right\}. \quad (1)$$

Such modification does not assume the existence of a new dimensional parameter in the action.

Here $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2$,
the Planck mass being $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$ GeV,
g is the determinant of the metric tensor $g_{\mu\nu}$,
f a differentiable function,
Λ is the cosmological constant,
$\mathcal{L}_{\mathrm{matter}}$ is the matter Lagrangian,
$\Box$ is covariant d'Alembertian for a scalar field

$$\Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right).$$

For FLRW geometry the d'Alembertian has the form:

$$\Box = -(\partial_t^2 + 3H\partial_t), \text{ where } H = \frac{\dot{a}}{a} \text{ is the Hubble parameter}$$

the its inverse operator (see [2]) is :

$$\Box^{-1} = -\int_0 dt' \frac{1}{a^3(t')} \int_0^{t'} dt'' a^3(t'')$$

$$\Box^{-1} R = -\int_0 dt' \frac{1}{a^3(t')} \int_0^{t'} dt'' a^3(t'') \left[12H^2(t'') + 6\dot{H}^2(t'')) \right].$$

[2] N.C. Tsamis, R.P. Woodard, CCTP-2010-8, UFIFT-QG-10-04, `arXiv:1006.4834[gr-qc]`

This nonlocal model has a local scalar-tensor formulation.
Introducing two scalar fields, η and ξ, we rewrite action (1) in local form:

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\eta)-\xi\right)+\xi\Box\eta-2\Lambda\right]+\mathcal{L}_{\mathrm{matter}}\right\}. \tag{2}$$

By varying the action (2) over ξ, we get $\Box\eta=R$.
Substituting $\eta=\Box^{-1}R$ into action (2), one reobtains action (1).
Varying action (2) with respect to the metric tensor $g_{\mu\nu}$, one gets

$$\frac{1}{2}g_{\mu\nu}\left[R\left(1+f(\eta)-\xi\right)-\partial_\rho\xi\partial^\rho\eta-2\Lambda\right]-R_{\mu\nu}\left(1+f(\eta)-\xi\right)+ \tag{3}$$

$$+\frac{1}{2}\left(\partial_\mu\xi\partial_\nu\eta+\partial_\mu\eta\partial_\nu\xi\right)-\left(g_{\mu\nu}\Box-\nabla_\mu\partial_\nu\right)\left(f(\eta)-\xi\right)+\kappa^2T_{\mathrm{matter}\,\mu\nu}=0,$$

where ∇_μ is the covariant derivative, $T_{\mathrm{matter}\,\mu\nu}$ is the energy–momentum tensor of matter.

Variation of action (2) with respect to η yields

$$\Box\xi+f'(\eta)R=0. \tag{4}$$

In the spatially flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right) \tag{5}$$

the system of Eqs. (3) is equivalent to the following equations:

$$-3H^2(1 + f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\frac{d}{dt}(f(\eta) - \xi) + \Lambda + \kappa^2\rho_{\rm m} = 0\,, \tag{6}$$

$$(2\dot{H}+3H^2)(1+f(\eta)-\xi)+\frac{1}{2}\dot{\xi}\dot{\eta}+\left(\frac{d^2}{dt^2}+2H\frac{d}{dt}\right)(f(\eta)-\xi)-\Lambda+\kappa^2P_{\rm m} = 0. \tag{7}$$

For a perfect matter fluid, we have $T_{\mathrm{matter}\,00} = \rho_{\mathrm{m}}$, $T_{\mathrm{matter}\,ij} = P_{\mathrm{m}} g_{ij}$. The equation of state (EoS) is

$$\dot{\rho}_{\mathrm{m}} = -3H(P_{\mathrm{m}} + \rho_{\mathrm{m}}). \tag{8}$$

Adding up Eqs. (6) and (7), we get

$$2\dot{H}\left(1 + f(\eta) - \xi\right) + \dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} - H\frac{d}{dt}\right)\left(f(\eta) - \xi\right) + \kappa^2(P_{\mathrm{m}} + \rho_{\mathrm{m}}) = 0. \tag{9}$$

The equations of motion for the scalar fields η and ξ follow

$$\ddot{\eta} + 3H\dot{\eta} = -6\left(\dot{H} + 2H^2\right), \tag{10}$$

$$\ddot{\xi} + 3H\dot{\xi} = 6\left(\dot{H} + 2H^2\right) f'(\eta), \tag{11}$$

where we have used $R = 6\dot{H} + 12H^2$.

The system of equations (8)–(11) together with (6) is equivalent to the full system of Einstein's equations.

Nonlocal models with de Sitter solutions

Assuming that the Hubble parameter is a nonzero constant: $H = H_0$ we obtain

$$\eta(t) = \; -4H_0(t-t_0) - \eta_0 e^{-3H_0(t-t_0)},$$

t_0, η_0 are integration constants. Without loss of generality we set $t_0 = 0$.

Considering $w_{\mathrm{m}} \equiv P_{\mathrm{m}}/\rho_{\mathrm{m}} = \mathit{const} \neq -1$ we obtain from Eq. (8)

$$\rho_{\mathrm{m}} = \rho_0 \, e^{-3(1+w_{\mathrm{m}})H_0 t}, \qquad (12)$$

where ρ_0 is an arbitrary constant.

From Einstein equations we obtain linear differential equation to $\Psi(t) = f(\eta(t)) - \xi(t)$:

$$\ddot{\Psi} + 5H_0\dot{\Psi} + 6H_0^2(1+\Psi) - 2\Lambda + \kappa^2(w_{\mathrm{m}} - 1)\rho_{\mathrm{m}} = 0, \qquad (13)$$

Equation (13) has the following general solution:

- At $\rho_0 = 0$,

$$\Psi_1(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2}, \tag{14}$$

- At $\rho_0 \neq 0$ and $w_{\rm m} = 0$,

$$\Psi_2(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2 \rho_0}{H_0} e^{-3H_0 t} t, \tag{15}$$

- At $\rho_0 \neq 0$ and $w_{\rm m} = -1/3$,

$$\Psi_3(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} + \frac{4\kappa^2 \rho_0}{3H_0} e^{-2H_0 t} t, \tag{16}$$

- At $\rho_0 \neq 0$, $w_{\rm m} \neq 0$ and $w_{\rm m} \neq -1/3$,

$$\Psi_4(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2 \rho_0 (w_{\rm m} - 1) e^{-3H_0 (w_{\rm m}+1) t}}{3H_0^2 w_{\rm m} (1 + 3w_{\rm m})}. \tag{17}$$

Substituting $\xi(t) = f(\eta(t)) - \Psi(t)$ into

$$\Box\xi + f'(\eta)R = 0 \tag{18}$$

we get linear differential equation to $f(\eta)$:

$$\dot{\eta}^2 f''(\eta) + \left(\ddot{\eta} + 3H_0\dot{\eta} - 12H_0^2\right) f'(\eta) = \ddot{\Psi} + 3H_0\dot{\Psi}\,. \tag{19}$$

Therefore, the model, which is described by action (2), can have de Sitter solutions only if $f(\eta)$ satisfies Eq. (19). In other words Eq. (19) is a necessary condition that the model has de Sitter solutions. To prove the existence of de Sitter solutions for the given $f(\eta)$ one should also check Eqs. (6) and (7).

To demonstrate how one can get $f(\eta)$, which admits the existence of de Sitter solutions, in the explicit form, we restrict ourselves to the case $\eta_0 = 0$. In this case, Eq. (19) has the following form:

$$16H_0^2 f''(\eta) - 24H_0^2 f'(\eta) = \Phi(\eta), \tag{20}$$

where $\Phi(\eta) = \Phi(-4H_0 t) \equiv \ddot{\Psi} + 3H_0\dot{\Psi}$.

Substituting the explicit form of $\Psi(t)$ to (19), we get

- For the model without matter ($\rho_0 = 0$, $\Psi(t) = \Psi_1(t)$),

$$f_1(\eta) = \frac{C_2}{4}e^{\eta/2} + C_3 e^{3\eta/2} + C_4. \tag{21}$$

- For the model with the dark matter ($w_{\rm m} = 0$, $\Psi(t) = \Psi_2(t)$),

$$f_2(\eta) = f_1(\eta) - \frac{\kappa^2\rho_0}{3H_0^2}e^{3\eta/4}. \tag{22}$$

- For the model, including the matter with $w_{\rm m} = -1/3$ ($\Psi(t) = \Psi_3(t)$),

$$f_3(\eta) = f_1(\eta) + \frac{\kappa^2\rho_0}{4H_0^2}\left(1 - \frac{1}{3}\eta\right)e^{\eta/2}. \tag{23}$$

- For the model, including the matter with another value of $w_{\rm m}$ ($\Psi(t) = \Psi_4(t)$),

$$f_4(\eta) = f_1(\eta) - \frac{\kappa^2\rho_0}{3(1+3w_{\rm m})H_0^2}e^{3(w_{\rm m}+1)\eta/4}. \tag{24}$$

De Sitter solutions for exponential $f(\eta)$

One can see that the key ingredient of all functions $f_i(\eta)$ is an exponent function. In the following we consider de Sitter solutions for the model with

$$f(\eta) = f_0 e^{\eta/\beta}\,, \tag{25}$$

where f_0 and β are constant.

The model of exponential function $f(\eta)$ is actively studied in
S. Nojiri and S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821;
S. Jhingan, S. Nojiri, S.D. Odintsov, M. Sami, I. Thongkool, and S. Zerbini, *Phys. Lett.* B **663** (2008) 424–428 ;
T.S. Koivisto, *Phys. Rev.* D **77** (2008) 123513;
S. Nojiri, S.D. Odintsov, M. Sasaki and Y.l. Zhang, *Phys. Lett.* B **696** (2011) 278–282;
Y.l. Zhang and M. Sasaki, *Int. J. Mod. Phys.* D **21** (2012) 1250006.

De Sitter solutions play a very important role in cosmological models, because both inflation and the late-time Universe acceleration can be described as a de Sitter solution with perturbations. A few de Sitter solutions for this model have been found in [3] and also analyzed in [4].

We generalize de Sitter solutions from K. Bamba, Sh. Nojiri, S.D. Odintsov, and M. Sasaki, YITP-11-46, arXiv:1104.2692 without any restriction on parameters.

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[3] S. Nojiri and S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821

[4] K. Bamba, Sh. Nojiri, S.D. Odintsov, and M. Sasaki, *Screening of cosmological constant for De Sitter Universe in non-local gravity, phantom-divide crossing and finite-time future singularities*, YITP-11-46, `arXiv:1104.2692`

For $\beta \neq 4/3$, from (10) and (11) the following solution is obtained:

$$\xi = -\frac{3f_0\beta}{3\beta-4}e^{-4H_0(t-t_0)/\beta} + \frac{c_0}{3H_0}e^{-3H_0(t-t_0)} - \xi_0, \tag{26}$$

$$\eta = -4H_0(t-t_0),\ c_0 \text{ is an arbitrary constant,}$$

$$\Lambda = 3H_0^2(1+\xi_0), \qquad \rho_0 = \frac{6\,(\beta-2)\,H_0^2 f_0}{\kappa^2\beta}, \qquad w_{\mathrm{m}} = -1+\frac{4}{3\beta}. \tag{27}$$

For $\beta = 4/3$, we get

$$\xi(t) = -f_0(c_0+3H_0(t-t_0))e^{-3H_0(t-t_0)} - \xi_0, \tag{28}$$

$$\Lambda = 3H_0^2(1+\xi_0), \qquad P_{\mathrm{m}} = 0, \qquad \rho_{\mathrm{m}} = -\frac{3}{\kappa^2}H_0^2 f_0 e^{-3H_0(t-t_0)}. \tag{29}$$

This solution clearly corresponds to dark matter, because $w_{\mathrm{m}} = 0$.

Stability of the de Sitter background

The case of nonzero Λ, the FLRW metric

Let us now introduce new variables

$$\phi = f(\eta) = f_0 e^{\eta/\beta}, \qquad \psi = \dot{\eta}, \qquad \dot{\vartheta} = \xi. \tag{30}$$

The functions $\phi(t)$ and $\psi(t)$ are connected by the equation

$$\dot{\phi} = \frac{1}{\beta}\phi\psi. \tag{31}$$

Consider the de Sitter solution

$$\rho_{\rm m} = \rho_0 e^{-3(w_{\rm m}+1)H_0(t-t_0)}, \quad P_{\rm m} = w_{\rm m}\rho_{\rm m}, \quad \Lambda = 3H_0^2(1+\xi_0),$$
$$\beta = \frac{4}{3(1+w_{\rm m})}, \quad \psi = -4H_0, \quad \phi = f_0 e^{-4H_0 t/\beta}. \tag{32}$$

For $\beta \neq 4/3$, we have

$$\xi = -\frac{3f_0\beta}{3\beta - 4}e^{-4H_0(t-t_0)/\beta} + \frac{c_0}{3H_0}e^{-3H_0(t-t_0)} - \xi_0,$$

and, for $\beta = 4/3$,

$$\xi = -f_0(c_0 + 3H_0(t - t_0))e^{-3H_0(t-t_0)} - \xi_0.$$

As t tends to $+\infty$,

$$\rho_{\mathrm{m}} \to 0, \qquad \phi \to 0, \qquad \psi = -4H_0, \qquad \xi \to -\xi_0, \tag{33}$$

for all $H_0 > 0$ and $\beta > 0$. This system has a fixed point:

$$\phi = 0, \qquad \xi = -\xi_0, \qquad \psi = -4H_0, \qquad \rho_{\mathrm{m}} = 0.$$

In the neighborhood of the fixed point, which corresponds to de Sitter solution, we have

$$
\begin{aligned}
H(t) &= H_0 + \varepsilon h_1(t) + \mathcal{O}(\varepsilon^2),\\
\phi(t) &= \varepsilon \phi_1(t) + \mathcal{O}(\varepsilon^2),\\
\psi(t) &= -4H_0 + \varepsilon \psi_1(t) + \mathcal{O}(\varepsilon^2),\\
\xi(t) &= -\xi_0 + \varepsilon \xi_1(t) + \mathcal{O}(\varepsilon^2),\\
\vartheta(t) &= \varepsilon \vartheta_1(t) + \mathcal{O}(\varepsilon^2),\\
\rho_{\mathrm{m}}(t) &= \varepsilon \rho_{m1}(t) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

where ε is a small parameter.

From system of Einstein's equations we obtain the following:

$$\dot{\rho}_{m1} = -\frac{4}{\beta}H_0\rho_{m1}, \tag{34}$$

$$\dot{\phi}_1 = -\frac{4}{\beta}H_0\phi_1, \tag{35}$$

$$\dot{\vartheta}_1 = -3H_0\vartheta_1 + \frac{12}{\beta}H_0^2\phi_1, \tag{36}$$

$$\dot{h}_1 = \frac{2}{(1+\xi_0)}\left[\frac{2}{\beta}\left(1-\frac{2}{\beta}\right)H_0^2\phi_1 - \frac{\kappa^2}{3\beta}\rho_{m1}\right], \tag{37}$$

$$\dot{\psi}_1 = -3H_0\psi_1 - 12H_0h_1 - \frac{12}{(1+\xi_0)}\left[\frac{2}{\beta}\left(1-\frac{2}{\beta}\right)H_0^2\phi_1 - \frac{\kappa^2}{3\beta}\rho_{m1}\right]. \tag{38}$$

Note that the function ξ_1 is not included in this system. It can be defined using Eq. (6). It is plain that ξ_1 cannot tend to infinity, if all other first-order corrections are bounded.

Let us now consider the system (34)–(38). The functions

$$\rho_{m1}(t) = d_0 e^{-4H_0 t/\beta}, \quad \phi_1(t) = d_1 e^{-4H_0 t/\beta}, \tag{39}$$

where d_0, d_1 are arbitrary constants, are general solutions to (34), (35), respectively. Substitute these functions into the other equations:

$$h_1(t) = d_2 - \frac{6H_0^2 d_1(\beta - 2) - \kappa^2 d_0 \beta}{6\beta H_0(1+\xi_0)} e^{-4H_0 t/\beta}, \tag{40}$$

$$\vartheta_1(t) = 12\frac{H_0 d_1}{3\beta - 4} e^{-4H_0 t/\beta} + d_3 e^{-3H_0 t}, \tag{41}$$

$$\psi_1(t) = \frac{2(\beta-2)(6H_0^2\beta d_1 - 12H_0^2 d_1 - \kappa^2\beta d_0)}{H_0\beta(3\beta-4)(1+\xi_0)} e^{-4H_0 t/\beta} + d_4 e^{-3H_0 t} - 4d_2, \tag{42}$$

where d_2, d_3, and d_4 are arbitrary constants. The two last expressions are valid for $\beta \neq 4/3$.

For $\beta = 4/3$,

$$\vartheta_1 = \left(9H_0^2 d_1 t + d_3\right) e^{-3H_0 t},$$

$$\psi_1 = \left(\frac{(3H_0^2 d_1 + \kappa^2 d_0)t}{1+\xi_0} + d_4\right) e^{-3H_0 t} - 4d_2.$$

We see that none of the perturbations tends to infinity at $t \to \infty$ at $\beta > 0$ and $H_0 > 0$.

Thus, for $H_0 > 0$ and $\beta > 0$, the de Sitter solutions are stable with respect to fluctuations of the initial conditions in the FLRW metric at any nonzero value of Λ.

The case of nonzero Λ, the Bianchi I metric

The Bianchi universe models are spatially homogeneous anisotropic cosmological models. Interpreting the solutions of the Friedmann equations as isotropic solutions in the Bianchi I metric, we include anisotropic perturbations in our consideration. The stability analysis is essentially simplified by a suitable choice of variables. Let us consider the Bianchi I metric

$$ds^2 = -dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2. \tag{43}$$

$$a_i(t) = a(t)e^{\beta_i(t)}. \tag{44}$$

Imposing the constraint $\beta_1(t)+\beta_2(t)+\beta_3(t)=0$, at any t, one has the following relations

$$a(t)=\left[a_1(t)a_2(t)a_3(t)\right]^{1/3}, \quad H_i\equiv\frac{\dot a_i}{a_i}=H+\dot\beta_i, \tag{45}$$

$$H\equiv\frac{\dot a}{a}=\frac{1}{3}(H_1+H_2+H_3). \tag{46}$$

In the case of the FLRW spatially flat metric we have $a_1=a_2=a_3=a$, all $\beta_i=0$, and H is the Hubble parameter. We introduce the shear

$$\sigma^2\equiv\dot\beta_1^2+\dot\beta_2^2+\dot\beta_3^2. \tag{47}$$

In the Bianchi I metric $R=12H^2+6\dot H+\sigma^2$.

The field and Einstein equations system for the Bianchi I metric has a fixed point, corresponding to the de Sitter solution:

$$\phi=0, \qquad \xi=\,-\,\xi_0, \qquad \psi=\,-\,4H_0, \qquad \rho_{\rm m}=0, \qquad \sigma^2=0.$$

In the neighborhood of the fixed point we have

$$
\begin{aligned}
H(t) &= H_0 + \varepsilon h_1(t) + \mathcal{O}(\varepsilon^2),\\
\phi(t) &= \varepsilon \phi_1(t) + \mathcal{O}(\varepsilon^2),\\
\psi(t) &= -4H_0 + \varepsilon \psi_1(t) + \mathcal{O}(\varepsilon^2),\\
\xi(t) &= -\xi_0 + \varepsilon \xi_1(t) + \mathcal{O}(\varepsilon^2),\\
\vartheta(t) &= \varepsilon \vartheta_1(t) + \mathcal{O}(\varepsilon^2),\\
\rho_{\mathrm{m}}(t) &= \varepsilon \rho_{m1}(t) + \mathcal{O}(\varepsilon^2),\\
\sigma^2(t) &= \varepsilon \sigma_1^2(t) + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

where ε is a small parameter.

From fields and Einstein equations we obtain $\sigma_1^2 = d_5 e^{-6H_0 t}$,

$$h_1 = d_2 - \frac{6H_0^2 d_1(\beta - 2) - \kappa^2 d_0 \beta}{6\beta H_0(1+\xi_0)} e^{-4H_0 t/\beta} + \frac{d_5}{12H_0} e^{-6H_0 t}, \tag{48}$$

$$\rho_{m1}(t) = d_0 e^{-4H_0 t/\beta}, \quad \phi_1(t) = d_1 e^{-4H_0 t/\beta}, \tag{49}$$

$$\vartheta_1(t) = 12\frac{H_0 d_1}{3\beta - 4} e^{-4H_0 t/\beta} + d_3 e^{-3H_0 t}. \tag{50}$$

$$\begin{aligned} \psi_1 \; &= \; \frac{2(\beta-2)(6H_0^2\beta d_1 - 12H_0^2 d_1 - \kappa^2\beta d_0) e^{\frac{-4H_0 t}{\beta}}}{H_0\beta(3\beta-4)(1+\xi_0)} + d_4 e^{-3H_0 t} - \\ &- \; 4d_2 - \frac{d_5 e^{-6H_0 t}}{3H_0}, \quad \beta \neq 4/3 \end{aligned} \tag{51}$$

$$\psi_1 = \left(\frac{\left(3H_0^2 d_1 + \kappa^2 d_0\right) t}{1+\xi_0} + d_4 \right) e^{-3H_0 t} - 4d_2 - \frac{d_5}{3H_0} e^{-6H_0 t}, \quad \beta = 4/3$$

Thus, for $H_0 > 0$ and $\beta > 0$, the de Sitter solutions are stable with respect to fluctuations of the initial conditions in the Bianchi I metric at any nonzero value of Λ.

The stability of de Sitter solutions with respect to fluctuations of the initial conditions in the Bianchi I metric, in the case $\Lambda = 0$.

To analyze the stability of the de Sitter solutions at $\Lambda = 0$, we have considered the system of equations using the Hubble-normalized variables

$$X = -\frac{\dot{\eta}}{4H}, \quad W = \frac{\dot{\xi}}{6Hf}, \quad Y = \frac{1-\xi}{3f}, \quad Z = \frac{\kappa^2\rho_m}{3H^2 f}, \quad K = \frac{\sigma^2}{2H^2}$$

and the independent variable, N,

$$\frac{d}{dN} \equiv a\frac{d}{da} = \frac{1}{H}\frac{d}{dt} \ .$$

The use of these variables makes the equation of motion dimensionless.

Field and Einstein equations in terms of the new variables have the fixed point

$$H = H_0, \quad X_0 = 1, \quad Z_0 = \frac{2(\beta - 2)}{\beta}, \quad W_0 = \frac{2}{3\beta - 4}, \quad K_0 = 0,$$

which corresponds to de Sitter solution for $\beta \neq 4/3$, with $c_0 = 0$. In the case of an arbitrary c_0, for the de Sitter solution, we get

$$W = \frac{2}{3\beta - 4} - \frac{c_0}{6H_0 f_0} e^{-(3-4/\beta)(N-N_0)},$$

where $N_0 = H_0 t_0$. The function W tends to infinity at large N for $\beta < 4/3$ and $\lim\limits_{N\to\infty} W = W_0$ at $\beta > 4/3$.

So, the fixed point can be stable only at $\beta > 4/3$.

The consideration of perturbations in the neighborhood of the fixed point shows that the perturbations decrease at $4/3 < \beta \leqslant 2$ at $H_0 > 0$. Thus, the de Sitter solutions are stable with respect to perturbations of the Bianchi I metric, in the case $4/3 < \beta \leqslant 2$ at $H_0 > 0$.

Conclusion

- A nonlocal gravity model with a function $f(\Box^{-1}R)$ has been considered and it has been proved that this model has de Sitter solutions only if the function f satisfies the second-order linear differential equation (19).

- The de Sitter solutions have been obtained in the most general form and their stability in the FLRW and Bianchi I metric has been analysed.

- The de Sitter solution is stable both for $\Lambda > 0$, and for $\Lambda < 0$. So, it is possible that the cosmological constant is negative, but due to nonlocality we get stable de Sitter solution at $H_0 > 0$.

- The stability conditions in the cases of the FLRW and Bianchi I metrics coincide.