Generalized superstatistics, branching processes, and pair production in a neutron star magnetosphere

Denis Sob'yanin

Tamm Department of Theoretical Physics,
Lebedev Physical Institute, Russian Academy of Sciences, Moscow, Russia

Phys. Rev. E **84**, 051128 (2011)

Generalized superstatistics

- a new approach to the study of complex nonequilibrium systems (Sob'yanin 2011)
- a generalization of superstatistics

Nonequilibrium systems

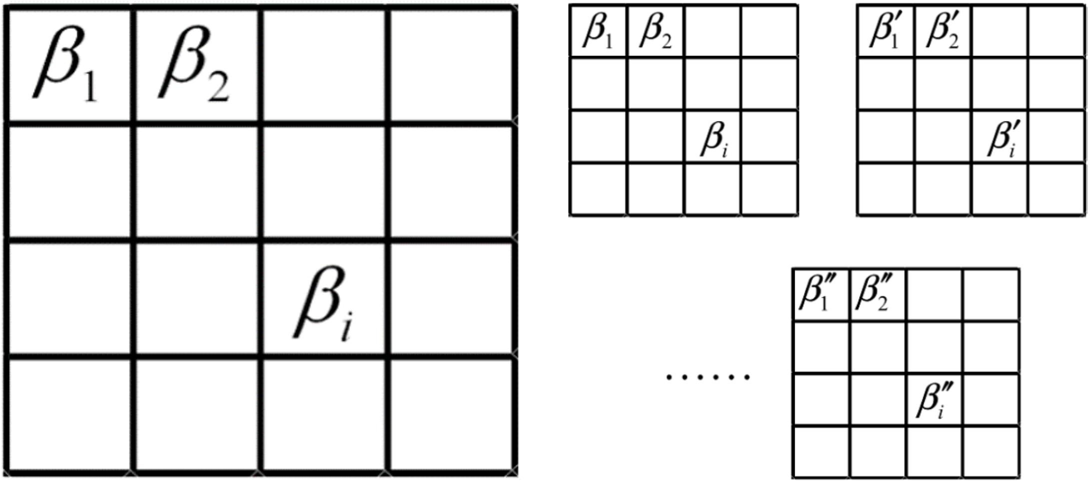
- Exhibit spatiotemporally inhomogeneous dynamics.
- Often characterized by hierarchical structures of dynamics.
- The hierarchy is formed by the decomposition of the system dynamics into different dynamics on different spatiotemporal scales.
- The statistical properties of the system can be effectively described by a superposition of several statistics.

Superstatistics

- Formulated to consider nonequilibrium systems with a stationary state and intensive parameter fluctuations (Beck & Cohen 2003).
- Superstatistical systems are characterized by the existence of an intensive parameter β.
- An essential feature is sufficient time scale separation between two relevant dynamics within the complex system.
- β fluctuates on a much larger time scale than the typical relaxation time of the local dynamics.
- A superstatistical system can be thought of as a collection of many small spatial cells, each having the Gibbs canonical distribution determined by β.
- β is often the inverse temperature in a cell, but other interpretations are possible.

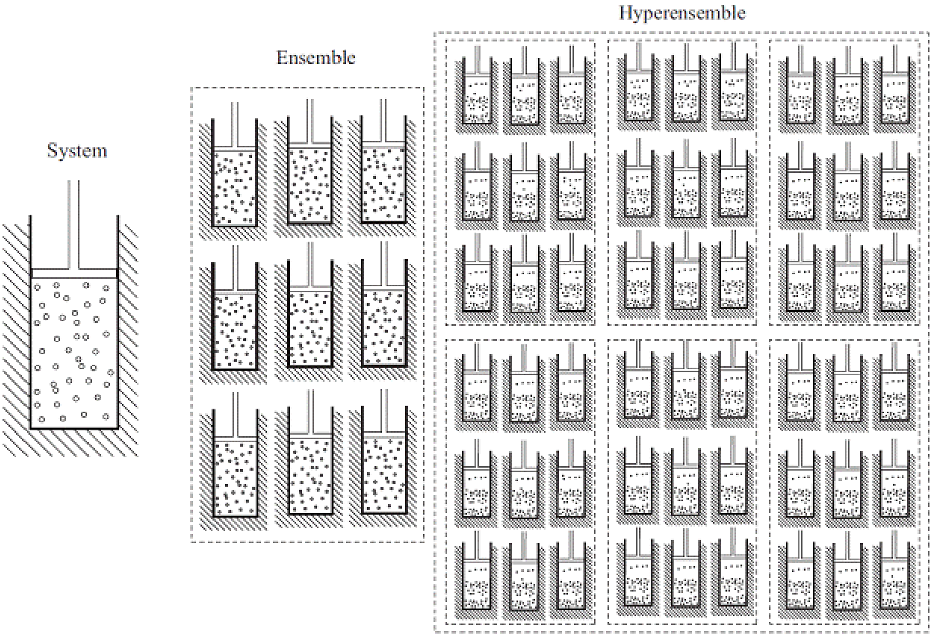
Superstatistical system

- can be associated with a hyperensemble (Abe 2009)



Hyperensemble

- an ensemble of ensembles (Crooks 2007)



Applications of superstatistics

- Astrophysical
  - cosmic-ray energy spectra and electron-positron pair annihilation (Beck 2004, 2009)
  - solar flares (Baiesi, Paczuski, & Stella 2006)
- and many others
  - random matrix theory (Abul-Magd 2006)
  - multiplicative noise (Duarte Queirós 2008)
  - Feynman propagators (Jizba & Kleinert 2010)
  - nonequilibrium Markovian systems (Lubashevsky et al. 2009)
  - system lifetime distributions (Ryazanov 2009)
  - a mesoscopic approach to Brownian motion (Rodriguez & Santamaria-Holek 2007)

Applications of superstatistics

- cancerous systems (Leon Chen & Beck 2008)
- complex networks (Abe & Thurner 2005)
- train departure delays (Briggs & Beck 2007)
- hydroclimatic fluctuations (Porporato, Vico, & Fay 2006)
- wind velocity fluctuations (Rizzo & Rapisarda 2004)
- share price fluctuations (Anteneodo & Duarte Queirós 2009; Van der Straeten & Beck 2009)
- hydrodynamic turbulence (Reynolds 2003; Jung & Swinney 2005; Beck, Cohen, & Swinney 2005; Beck 2007; Van der Straeten & Beck 2009; Abe 2010)
- etc.

Generalized superstatistics

- represents a "statistics of superstatistics"
- based on the concept of fluctuating control parameters
- can be used for nonstationary nonequilibrium systems

Generalized superstatistical system

- comprises a set of nonequilibrium superstatistical subsystems
- has three levels of dynamics:
  - fast local dynamics in a cell
  - superstatistical dynamics in a subsystem
  - global dynamics in the whole system
- can be associated with a generalized hyperensemble, an ensemble of hyper-ensembles

Generalized superstatistics

- There exists a fluctuating vector control parameter ξ on which both the intensive parameter distribution and the density of energy states depend.
- ξ determines the density of energy states for the subsystem,

$$g(E|\xi) = \frac{\partial \Gamma(E|\xi)}{\partial E},$$

  where $\Gamma(E|\xi)$ is the number of states with energy less than E.
- The Gibbs canonical distribution for each cell of the subsystem is

$$\rho_G(E|\beta, \xi) = \frac{e^{-\beta E}}{Z(\beta|\xi)},$$

  where

$$Z(\beta|\xi) = \int e^{-\beta E} d\Gamma(E|\xi)$$

  is the partition function.

Generalized superstatistical distribution

- ξ also determines the distribution $f(\beta|\xi)$ of the intensive parameter β.
- The superstatistical distribution for each subsystem is given by

$$\rho(E|\xi) = \int \rho_G(E|\beta,\xi) f(\beta|\xi) d\beta,$$

  with the normalization condition $\int \rho(E|\xi) d\Gamma(E|\xi) = 1$.
- The generalized superstatistical distribution has the form

$$\sigma(E) = \int \rho(E|\xi) g(E|\xi) c(\xi) d\xi,$$

  with the normalization condition $\int \sigma(E) dE = 1$.

An example: branching processes

- Consider a many-particle system composed of particles of n types. Each type-i particle (T_i) has a random lifetime with a probability distribution function $G_i(\tau)$.
- At the end of its life the particle decays into a random number of particles of several types.
- Specifically, at the moment of its decay the particle produces $\omega_j \geqslant 0$ type-j particles of age zero, $1 \leqslant j \leqslant n$:

$$T_i \to \sum_{j=1}^{n} \omega_j T_j.$$

- We have a multitype age-dependent branching process, the so-called multitype Sevast'yanov process (Sevast'yanov 1964).

Physical assumptions

- The mean number of type-j particles that appear upon the decay of a type-i particle is given by an $n \times n$ matrix $A = \|A_{ij}\|$ with components $0 \leqslant A_{ij} < \infty$.
- A is irreducible, or indecomposable, i.e., the index set $\{1, \ldots, n\}$ cannot be divided into two disjoint nonempty sets S_1 and S_2 such that $A_{ij} = 0$ for all $i \in S_1$ and all $j \in S_2$.
- The Perron root of A, i.e., the maximum positive real eigenvalue of A, is greater than one.
- We deal with the indecomposable supercritical multitype age-dependent branching process.
- Physically, this means that
    - a particle of a given type potentially has descendants, either direct or distant, of any type and
    - the number of particles in the system, on average, progressively increases.

Long-run properties

- The mean number of particles of any type at time t is

$$\propto e^{\alpha t}, \qquad t \to \infty.$$

- The limiting probability π_i that a given particle is of type i is independent of the type of the primary particle.
- Nonstationary though the situation is, the limiting probability is stationary.
- The limiting age distribution for type-i particles is (Sob'yanin 2011)

$$L_i(\tau) = \frac{\int_0^\tau e^{-\alpha u}[1 - G_i(u)]du}{\int_0^\infty e^{-\alpha u}[1 - G_i(u)]du}.$$

- The energy of a type-i particle of age τ can be considered as a random variable characterized by a conditional probability density $w_i(E|\tau)$.
- The energy probability density for type-i particles becomes

$$\rho_i(E) = \int_0^\infty w_i(E|\tau)dL_i(\tau).$$

Branching processes and generalized superstatistics

- The described system can be considered as a generalized superstatistical system.
- The whole system is composed of n subsystems, the ith subsystem comprising type-i particles.
- The subsystems interact with each other in the sense that the decay of a particle in one subsystem leads to the creation of particles in other subsystems.
- The number of particles both in the whole system and in each subsystem increases exponentially.
- We have a nonstationary nonequilibrium situation.

Intensive parameter and control parameter distributions

- The control parameter ξ is a discrete random variable that yields the number of the subsystem to which a randomly chosen particle belongs.
- ξ has the discrete probability distribution $\{\pi_1, \ldots, \pi_n\}$ and corresponds to the particle type.
- The distribution of the intensive parameter β for the ith subsystem is

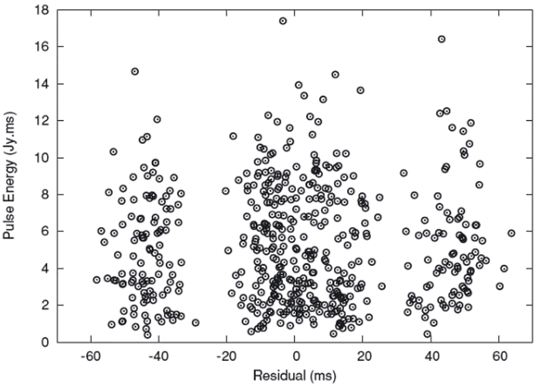
$$f_i(\beta) = Z_i(\beta)\mathfrak{L}^{-1}[\rho_i(E)](\beta),$$

where $\mathfrak{L}^{-1}[g(s)](x)$ is the inverse Laplace transform of a function $g(s)$ and $Z_i(\beta)$ is the partition function.

An astrophysical example: pair production in a neutron star magnetosphere

- New nonstationary cosmic radio sources associated with neutron stars:
  - intermittent pulsars (Kramer et al. 2006)
  - rotating radio transients (RRATs) (McLaughlin et al. 2006)
- Characteristic properties:
  - long "silence"
  - nonstationarity of radio emission
- An example: RRAT J1819–1458
  - period ≈ 4.263 s
  - burst rate $\sim 20 - 30\ \mathrm{h}^{-1}$
  - burst width ~ 3 ms

(Lyne et al. 2009)



Rotating radio transients (RRATs)

- Manifest themselves as separate, sparse, short, relatively bright radio bursts.
- The typical burst rate is from the range 1 min^{-1}–1 h^{-1}.
- The intensity of single radio bursts
  - reaches 310 mJy at 111 MHz (Shitov et al. 2009);
  - lies within the range of 100 mJy to 10 Jy at 1.4 GHz (Keane et al. 2010).
- The phase of bursts is approximately retained.
- The underlying periodicity lies within the range 0.1–6.7 s (Keane et al. 2010).
- For RRAT J1819–1458, the surface magnetic field reaches 5×10^{13} G (McLaughlin et al. 2006; Esamdin et al. 2008) and exceeds the critical one.
- The nature of RRATs can be explained by the formation of "lightnings" in their magnetospheres (Istomin & Sob'yanin 2011c).

Nonstationary pair production

- An electron-positron plasma outflowing from the magnetosphere of a neutron star is responsible for the observable radio emission.
- The plasma generation can be switched off for some time.
- The absorption of a high-energy photon in the inner neutron star magnetosphere triggers nonstationary cascade pair production (Istomin & Sob'yanin 2011a).
- This results in the formation of a "lightning" (Istomin & Sob'yanin 2011b).
- The plasma generation, along with the accompanying radio emission, is not suppressed even in ultrahigh magnetar magnetic fields (Istomin & Sob'yanin 2007, 2008).
- The properties of the emission from electrons and positrons are determined by their energies.
- It is important to find the energy distribution of particles.

Acceleration of particles

- The energy of a charged particle is characterized by its Lorentz factor $\gamma(\tau)$.
- The particle is efficiently accelerated by a longitudinal electric field $E_{\parallel}$.
- $\gamma(\tau)$ eventually reaches a stationary value γ_0, which is $\sim 10^8$ in a vacuum neutron star magnetosphere (Istomin & Sob'yanin 2009).
- At the initial stage of acceleration $\gamma(\tau)$ increases linearly with time,

$$\gamma(\tau) \approx E_{\parallel}\tau.$$

- When t approaches

$$\tau_0 = \gamma_0 / E_{\parallel},$$

  the radiation forces come to the fore.
- A need arises to use the Dirac-Lorentz equation to consider the particle dynamics properly (Istomin & Sob'yanin 2009, 2010a, 2010b).

Two types of particles

- A type-1 particle
  - can be efficiently accelerated by the electric field since the radiation friction is negligible
  - does not efficiently produce secondary pairs
- A type-2 particle
  - is not accelerated by the electric field because of the electrodynamic self-action effects
  - has the constant Lorentz factor γ_0
  - produces secondary pairs at a rate Q (Istomin & Sob'yanin 2011a)
- The particles of each produced pair, though moving independently of each other, can conveniently be considered as a whole.
- Type-1 and type-2 pairs are defined by analogy with individual particles.

Pair production and branching processes

- The Lorentz factors of type-1 and type-2 particles as functions of their ages become

$$\gamma_1(\tau) = E_{\|}\tau, \qquad 0 \leqslant \tau < \tau_0,$$
$$\gamma_2(\tau) = \gamma_0, \qquad 0 \leqslant \tau < \infty.$$

- The transformations of electron-positron pairs are

$$\begin{aligned} T_1 &\rightarrow T_2, \\ T_2 &\rightarrow T_1 + T_2. \end{aligned}$$

- The lifetime distribution functions are

$$\begin{aligned} G_1(\tau) &= \theta(\tau - \tau_0), \\ G_2(\tau) &= 1 - e^{-2Q\tau}, \end{aligned}$$

where $\theta(x)$ is the Heaviside function.

Pair production rate

- The mean matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ indicates that the branching process is supercritical and indecomposable.
- Pair production in the system under consideration is asymptotically described by the equation (Istomin & Sob'yanin 2011a)

$$\frac{dN(t)}{dt} = 2Q^{\text{eff}} N(t),$$

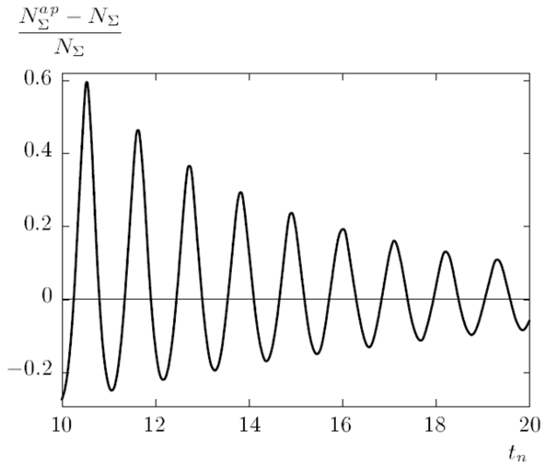
where $N(t)$ is the number of electron-positron pairs at time t, $Q^{\text{eff}} = N_{\tau_0}^{\text{eff}}/2\tau_0$ is the effective pair production rate, and $N_{\tau_0}^{\text{eff}}$ satisfies

$$N_{\tau_0}^{\text{eff}} = \ln N_{\tau_0} - \ln N_{\tau_0}^{\text{eff}},$$

with $N_{\tau_0} = 2Q\tau_0$.

- The Malthusian parameter is

$$\alpha = 2Q^{\text{eff}}.$$



Pair production and generalized superstatistics

- The system can be considered as a generalized superstatistical system.
- It consists of two superstatistical subsystems, the first comprising type-1 particles and the second comprising type-2 particles.
- The density of states for the subsystems is

$$\begin{aligned} g_1(\gamma) &= 1 - \theta(\gamma - \gamma_0), \\ g_2(\gamma) &= \delta(\gamma - \gamma_0), \end{aligned}$$

  where $\delta(x)$ is the delta function.
- The corresponding intensive parameter distributions are

$$\begin{aligned} f_1(\beta) &= \delta\Big(\beta - \frac{\alpha}{E_{\|}}\Big), \\ f_2(\beta) &= \delta(\beta). \end{aligned}$$

Pair production and generalized superstatistics

- The control parameter ξ corresponds to the type of a randomly chosen particle.
- The probability π_ξ that a randomly chosen particle is of type ξ is

$$\begin{aligned} \pi_1 &= 1 - \frac{\alpha}{2Q}, \\ \pi_2 &= \frac{\alpha}{2Q}. \end{aligned}$$

- π_2 may be interpreted as the probability that the particle significantly contributes to pair production.
- The generalized superstatistical distribution

$$\sigma(\gamma) = \frac{\alpha}{2Q}\,\delta(\gamma - \gamma_0) + [1 - \theta(\gamma - \gamma_0)]\,\frac{\alpha}{E_\parallel} e^{-\alpha\gamma/E_\parallel}$$

represents the energy distribution of ultrarelativistic electrons and positrons.

Summary

- Generalized superstatistics has been proposed, which is a statistics of superstatistics.
- It appears in the case of fluctuating control parameters and can be considered in the framework of generalized hyperensembles.
- The system with branching processes is an example of a nonstationary generalized superstatistical system.
- For nonstationary pair production in a neutron star magnetosphere, this approach allows one to obtain
  - the energy distribution of ultrarelativistic electrons and positrons and
  - the probability that a randomly chosen particle significantly contributes to the production of secondary electron-positron pairs.